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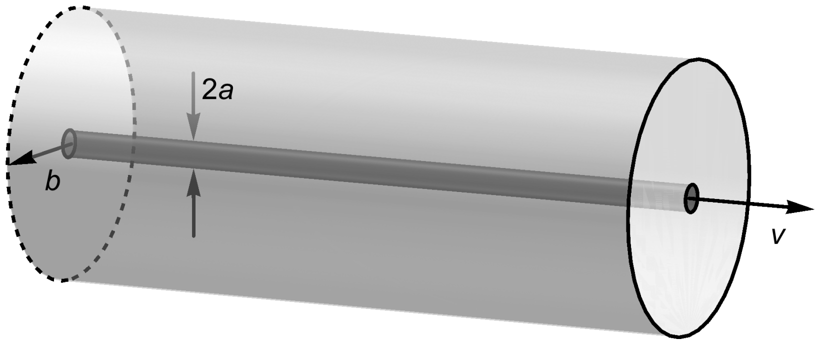
2. **Viscous flow**



A long cylindrical solid rod of radius a is at the center of and concentric with a long cylindrical pipe of inner radius b, with $b > a$. The region $a < r < b$ is filled with an incompressible viscous fluid of density ρ and viscosity μ, where r is the distance from the central axis of these cylinders. The solid rod is moving along the axis of the pipe with a small steady speed v, while the pipe stays at rest. Assume the fluid flow is in steady state with motion only parallel to the axis of the cylinders. Neglect gravity and assume that there is no gradient in the pressure P along the direction of the fluid motion. The Navier-Stokes equation for the velocity $\vec{u}(\vec{r})$ of the fluid is

$$\rho(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u}) = -\vec{\nabla} P + \mu \nabla^2 \vec{u} \ .$$

What is the steady state flow pattern $\vec{u}(\vec{r})$?

$$\rho\cancelto{0}{\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\right)} = -\cancelto{0}{\vec{\nabla} P} + \mu \nabla^2 \vec{u}$$

Steady state: convective derivative drops out

No pressure gradient: $\vec{\nabla} P = 0$

$$\nabla^2 \vec{u} = 0 \begin{cases} \nabla^2 u_r = 0, \text{ motion only parallel} \to u_r = 0 \\ \nabla^2 u_\phi = 0, \text{ motion only parallel} \to u_\phi = 0 \\ \nabla^2 u_z = 0, \text{ use solutions to laplace's eqn} \end{cases}$$

$$u_z(r,\phi) = A_0 + B_0 \ln r + \sum_{l=1}^{\infty} r^l \left(A_l \cos(l\phi) + B_l \sin(l\phi)\right) + r^{-l}\left(C_l \cos(l\phi) + D_l \sin(l\phi)\right)$$

$\downarrow \leftarrow$ axial symmetry: $A_l, B_l = 0$ for all l

$$= A_0 + B_0 \ln r$$

Boundary Conditions: $u_z(r=a) = V$, $u_z(r=b) = 0$

$$\left.\begin{aligned} 0 &= A_0 + B_0 \ln b \\ V &= A_0 + B_0 \ln a \end{aligned}\right\} \to B_0 = \frac{V}{\ln(a/b)}, \quad A_0 = -\frac{V}{\ln(a/b)} \ln b$$

$$u_z(r,\phi) = A_0 + B_0 \ln r = -\frac{V}{\ln(a/b)} \ln b + \frac{V}{\ln(a/b)} \ln r = V \frac{\ln(r/b)}{\ln(a/b)}$$

$$\boxed{\vec{u}(\vec{r}) = V \frac{\ln(r/b)}{\ln(a/b)} \hat{z}}$$