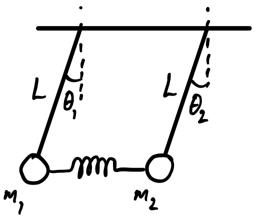
J17M.1 (Coupled Pendulums)

(a)




ē coordinates for this problem are $\theta_1$ & $\theta_2$.

ē posit$^{n}$ of mass 1 is: $\vec{r}_1 = \langle L\sin\theta_1, -L\cos\theta_1 \rangle$

ē posit$^{n}$ of mass 2 is: $\vec{r}_2 = \langle L\sin\theta_2, -L\cos\theta_2 \rangle$

ē Lagrangian is:

$$\mathcal{L} = T - V$$

$$= \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} + mgL\cos\theta_1 + mgL\cos\theta_2 - \frac{1}{2}\kappa(\theta_2 - \theta_1)^2$$

$$= \frac{1}{2} m_1 L^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L^2 \dot{\theta}_2^2 + mgL\cos\theta_1 + mgL\cos\theta_2 - \frac{1}{2}\kappa(\theta_1^2 + \theta_2^2 - 2\theta_1\theta_2)$$

$$\approx \frac{1}{2} m_1 L^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L^2 \dot{\theta}_2^2 - \frac{1}{2} mgL\theta_1^2 - \frac{1}{2} m_2 gL\theta_2^2 - \frac{1}{2}\kappa(\theta_1^2 + \theta_2^2 - 2\theta_1\theta_2)$$

Let $\vec{\eta} = \begin{pmatrix}\theta_1 \\ \theta_2\end{pmatrix} e^{-i\omega t}$

$$\Longrightarrow \mathcal{L} = \frac{1}{2}\dot{\vec{\eta}}^{\,T} \underbrace{\begin{pmatrix} m_1 L^2 & 0 \\ 0 & m_2 L^2 \end{pmatrix}}_{M} \dot{\vec{\eta}} - \frac{1}{2}\vec{\eta}^{\,T} \underbrace{\begin{pmatrix} m_1 gL + \kappa & -\kappa \\ -\kappa & m_2 gL + \kappa \end{pmatrix}}_{K} \vec{\eta}$$

To find ē normal frequencies, we solve $\det(K - \omega^2 M) = 0$ for $\omega^2$.

$$\begin{vmatrix} m_1 gL + \kappa - \omega^2 m_1 L^2 & -\kappa \\ -\kappa & m_2 gL + \kappa - \omega^2 m_2 L^2 \end{vmatrix} = 0.$$

$$(m_1 gL + \kappa - \omega^2 m_1 L^2)(m_2 gL + \kappa - \omega^2 m_2 L^2) - \kappa^2 = 0$$

$$m_1 m_2 g^2 L^2 + \cancel{\kappa^2} + \omega^4 m_1 m_2 L^4 + m_1 gL\kappa + m_2 gL\kappa - m_1 m_2 gL^3\omega^2 - m_1 m_2 gL^3\omega^2 - m_1\kappa L^2\omega^2 - m_2\kappa L^2\omega^2 - \cancel{\kappa^2} = 0$$

$$\omega^4(m_1 m_2 L^4) + \omega^2(-2m_1 m_2 gL^3 - m_1\kappa L^2 - m_2\kappa L^2) + (m_1 m_2 g^2 L^2 + m_1 g\kappa L + m_2 g\kappa L) = 0$$

$$\omega^2 = \frac{1}{2m_1 m_2 L^4}\Big[2m_1 m_2 gL^3 + m_1\kappa L^2 + m_2\kappa L^2$$
$$\pm\big(\cancel{4m_1^2 m_2^2 g^2 L^6} + m_1^2\kappa^2 L^4 + m_2^2\kappa^2 L^4 + \cancel{4m_1^2 m_2 g\kappa L^5} + \cancel{4m_1 m_2^2 g\kappa L^5} + 2m_1 m_2\kappa^2 L^4$$
$$- \cancel{4m_1^2 m_2^2 g^2 L^6} - \cancel{4m_1^2 m_2 g\kappa L^5} - \cancel{4m_1 m_2^2 g\kappa L^5}\big)^{1/2}\Big]$$

$$= \frac{1}{2m_1 m_2 L^4}\left(2m_1 m_2 gL^3 + m_1\kappa L^2 + m_2\kappa L^2 \pm \sqrt{m_1^2\kappa^2 L^4 + 2m_1 m_2\kappa^2 L^4 + m_2^2\kappa^2 L^4}\right)$$

$$= \frac{1}{2m_1 m_2 L^4}\left[2m_1 m_2 gL^3 + m_1\kappa L^2 + m_2\kappa L^2 \pm (m_1\kappa L^2 + m_2\kappa L^2)\right]$$

$$\Longrightarrow \omega^2 = \frac{g}{L} + \frac{\kappa}{L^2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \quad \text{or} \quad \omega^2 = \frac{g}{L}$$

$\bar{c}$ mode correspond$^{g}$ to $\omega^2 = \frac{g}{L}$ must be th for which $\tau = 0$ ($\bar{c}$ pendulums are in phase).

Thus, this normal mode has $\vec{\eta} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

By orthogonality, $\bar{c}$ other mode is $\vec{\eta} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, where they are out of phase so $\bar{c}$ tors$^{n}$ spr$^{g}$ stores some energy.

(b) For either eigenvector, evolut$^{n}$ is trivial: $\vec{\eta}_1(t) = \vec{\eta}_1(0)\, e^{i\omega_1 t}$

Thus, for a given set of initial condit$^{n}$s: $\theta_1(0)$ & $\theta_2(0)$

$$\vec{v}(0) = \begin{pmatrix}\theta_1(0)\\ \theta_2(0)\end{pmatrix} = \frac{\theta_1(0)}{\sqrt{2}}(\vec{\eta}_1 + \vec{\eta}_2) + \frac{\theta_2(0)}{\sqrt{2}}(\vec{\eta}_1 - \vec{\eta}_2)$$

$$= \frac{1}{\sqrt{2}}(\theta_1(0) + \theta_2(0))\,\vec{\eta}_1 + \frac{1}{\sqrt{2}}(\theta_1(0) - \theta_2(0))\,\vec{\eta}_2$$

$$\Longrightarrow \vec{v}(t) = \begin{pmatrix}\theta_1(t)\\ \theta_2(t)\end{pmatrix} = \frac{1}{\sqrt{2}}(\theta_1(0) + \theta_2(0))\,\vec{\eta}_1\, e^{-i\omega_1 t} + \frac{1}{\sqrt{2}}(\theta_1(0) - \theta_2(0))\,\vec{\eta}_2\, e^{-i\omega_2 t}$$ //.