

Section A. Mechanics

1. Coupled pendulums

Two simple pendulums of equal length L but unequal masses $m_1 > m_2$ hang in a uniform gravitational field g near their equilibrium positions. Let the (small) angular displacements of the pendulums from their equilibrium positions be $\theta_1 \ll 1$ and $\theta_2 \ll 1$, respectively. They are coupled by a torsional spring which applies a torque

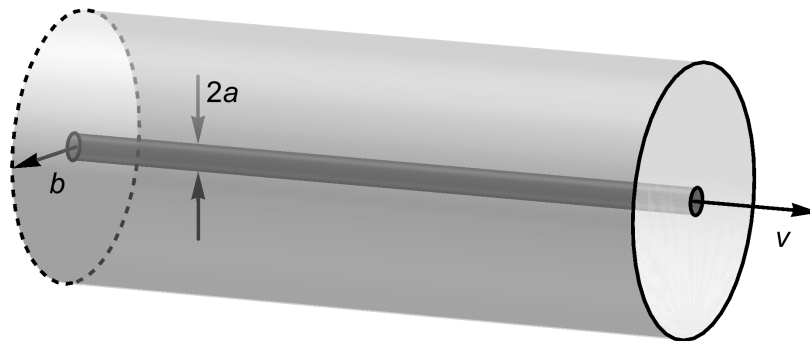
$$\tau_1 = \kappa(\theta_2 - \theta_1)$$

to pendulum 1 (and of course the opposite torque to pendulum 2). There is no damping.

(a) Solve for the normal modes of this system and their frequencies.

(b) If this system is started at time zero with initial small displacements $\theta_1(0)$ and $\theta_2(0)$, and both pendulums initially at rest, solve for the subsequent displacements $\theta_1(t)$ and $\theta_2(t)$ as functions of time t .

2. Viscous flow

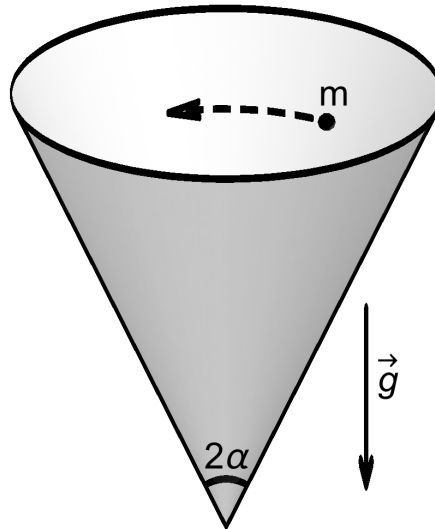


A long cylindrical solid rod of radius a is at the center of and concentric with a long cylindrical pipe of inner radius b , with $b > a$. The region $a < r < b$ is filled with an incompressible viscous fluid of density ρ and viscosity μ , where r is the distance from the central axis of these cylinders. The solid rod is moving along the axis of the pipe with a small steady speed v , while the pipe stays at rest. Assume the fluid flow is in steady state with motion only parallel to the axis of the cylinders. Neglect gravity and assume that there is no gradient in the pressure P along the direction of the fluid motion. The Navier-Stokes equation for the velocity $\vec{u}(\vec{r})$ of the fluid is

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right) = -\vec{\nabla} P + \mu \nabla^2 \vec{u}.$$

What is the steady state flow pattern $\vec{u}(\vec{r})$?

3. Orbits in a cone



A cone sits apex down with its axis vertical. The inside opening angle of the cone is 2α , with $\pi > 2\alpha > 0$, so the inside surface of the cone in cylindrical coordinates is $r = z \tan \alpha > 0$. A point mass slides without friction and nonrelativistically on the inside surface of the cone.

- (a) Solve for the circular orbit with radius r_0 . What is the period $T_0(r_0)$ of this orbit?
- (b) If this circular orbit is slightly perturbed away from circular, what is the period $T_1(r_0)$ of the resulting oscillations in the radial position r ? Show your work.

Section B. Electricity and Magnetism

1. Magnetic Levitation

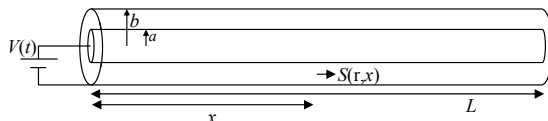
It is possible to make an object made from a diamagnetic material (such as bismuth or pyrolytic graphite) sit in equilibrium above a permanent magnet.

(a) Consider a small sphere of radius R made from a linear diamagnetic material with magnetic susceptibility $\chi_m < 0$, placed in a uniform magnetic field \vec{B}_0 . What is the total magnetic moment \vec{m} induced in the sphere? Note the direction of the magnetic moment.

(b) The sphere is placed at a height d ($d \gg R$) above the center of a horizontal circular wire loop of radius a ($a \gg R$) with current I flowing in the loop. Find the force on the sphere from the current loop and note its direction, for general a and d in this small R limit.

2. Coaxial cable

A long co-axial cable is connected to a power supply with a variable voltage $V(t)$. The cable has a cylindrical inner perfect conductor of outer radius a and a cylindrical outer perfect conductor with inner radius b . The length of the cable L is much greater than b . The space between the conductors is empty.

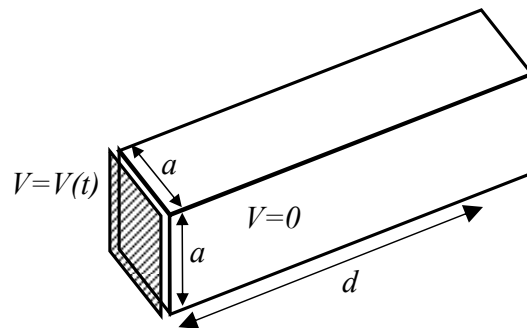


(a) The voltage $V(t)$ is very slowly ramped up from zero to a finite value V_0 (on a time scale much longer than L/c). There is nothing connected to the other end of the cable. Find the Poynting vector $\vec{S}(r, x)$ a distance x into the cable (far from the ends) and show that the time integral of the total energy flux at x is equal to the final electromagnetic energy stored from x to the end of the cable.

(b) Now the voltage is kept constant at V_0 and a resistor of resistance R is connected across the other end of the cable, so equal and opposite steady-state currents are flowing in the two conductors. Find the linear momentum stored in the cable and explain how a system with no net current can have a nonzero linear momentum.

3. Waveguide transmitter

A long square perfectly conducting waveguide with sides of width a is driven by a conductive plate near one of its ends as shown in the figure below. The plate is very close to the waveguide, but it is not touching its walls. The potential on the plate is given by $V(t) = V_0 \cos(\omega t)$ and is uniform across the surface of the plate. The interior of the waveguide is empty.



- (a) Find the range of frequencies ω for which only one mode excited by the plate will propagate in the waveguide.
- (b) If the frequency is in this range, find explicitly the electric field of the wave as a function of position in the waveguide at a distance d from the plate, for $d \gg a$.

Section A. Quantum Mechanics

1. Interacting spins

Four distinguishable spin-1/2 objects interact. The Hamiltonian is

$$H = A\vec{\sigma}_1 \cdot \vec{\sigma}_2 + B\vec{\sigma}_3 \cdot \vec{\sigma}_4 + C(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot (\vec{\sigma}_3 + \vec{\sigma}_4) ,$$

where $\vec{\sigma}_n$ are the Pauli spin operators for spin n .

- (a) List a complete set of good quantum numbers for the eigenstates of this Hamiltonian.
- (b) List the eigenenergies and the degeneracy of each energy level.
- (c) Using the basis of the eigenstates of the z -components of each spin, show a **ground state** for the case $A = B < 0 < C$. For this state give its complete set of good quantum numbers.

2. s-wave scattering resonances

Consider low-energy scattering of particles of mass m on a spherically symmetric, attractive square well potential: $V(r) = -V_0 < 0$ for $r \leq R$ and $V(r) = 0$ for $r > R$. The particles have a low fixed incident energy E , such that $E \ll \hbar^2/2mR^2$. Analyze the dependence of the total scattering cross-section σ on the depth of the potential V_0 . Make a sketch of the behavior of the cross-section $\sigma(V_0)$ as a function of V_0 , and calculate the location (in V_0) and height of any prominent features.

3. Perturbed Hydrogen Atom

A hydrogen atom interacts with a perturbing potential

$$\Delta V(x, y, z) = A(x^2 - y^2) ,$$

where (x, y, z) is the displacement of the electron from the nucleus. Assume $A > 0$. In all parts of this problem, ignore spins and relativistic effects.

(a) What must A be small compared to for perturbation theory to be a good approximation for the $n = 1$ and $n = 2$ eigenstates of this perturbed hydrogen atom?

(b) Give a rough estimate of the change in the ground state energy due to this perturbation, including the sign of the change. Your answer can leave a multiplicative factor that is positive and of order one undetermined.

(c) Describe qualitatively the shifts in the $n = 2$ energy levels at first order in perturbation theory. Which of the $n = 2$ levels, if any, are unshifted? What are the ratios of the different energy level shifts? What are the eigenstates of the shifted levels? Give a rough estimate of the energy shifts (again, you may leave a multiplicative factor that is positive and of order one undetermined).

Section B. Statistical Mechanics and Thermodynamics

1. Atomic/Molecular Gas

Consider a dilute gas of identical spinless atoms of mass m . They move nonrelativistically at temperature T . There is a very short range attraction between atoms that has only one bound state (a diatomic molecule) with binding energy Δ ; no molecules of more than two atoms are bound. The total number density of atoms including those in molecules is n .

Calculate the equilibrium equation of state for the pressure of this gas, assuming the gas remains dilute enough so that the motion of the atoms and molecules can be treated classically, although the binding of atoms to form molecules is necessarily quantum. Analyze your result in the two limits of: (a) weak, and (b) strong binding to show (and to check) that it agrees with simple arguments in these limits.

2. Liquid-Gas Critical Point

The van der Waals gas is a “simple” modification to the classical ideal gas. Each molecule is assumed to occupy a volume b , so the free volume available to a given molecule is reduced to $(V - Nb)$ for a gas of N molecules in a volume V . There is also an attractive interaction between the molecules, which lowers the energy of the gas, so the Helmholtz free energy of the van der Waals (*vdW*) gas is:

$$F_{vdW}(N, T, V) = F_{ideal}(N, T, V - Nb) - aN^2/V ,$$

where F_{ideal} is the free energy of a classical ideal gas, and $a > 0$ quantifies the attractive interaction.

- (a) What is the pressure $p(N, T, V)$ of this van der Waals gas?
- (b) If you know the equation of state $p(N, T, V)$ of a more general gas with a liquid-gas critical point, what calculation would you do to locate the critical point, T_c, p_c ?
- (c) Calculate the critical point T_c, p_c of this van der Waals gas.

3. Protein folding

The thermodynamics of protein folding plays an essential role in biology.

This problem is all at one atmosphere of pressure: Consider a large protein that folds at temperature T_1 , so for $T > T_1$ it is unfolded at equilibrium, while for a range of temperature T with $T < T_1$ it is folded at equilibrium in to a compact form and thus “hides” some hydrophobic parts of the long protein molecule from the surrounding water. Assume the folded (f) state of the protein has a temperature-independent specific heat $C^{(f)}$ and that its thermal expansion is negligible so $C_P^{(f)} = C_V^{(f)}$. Assume the same is true for the unfolded (u) state, so $C_P^{(u)} = C_V^{(u)} = C^{(u)}$ is also temperature-independent. The latent heat released on unfolding at temperature T_1 is $Q > 0$. The unfolded state has the larger specific heat: $\Delta C = C^{(u)} - C^{(f)} > 0$.

(a) What is the enthalpy ($H = E + PV$) difference, $\Delta H = H^{(u)} - H^{(f)}$, between the unfolded and folded states as a function of T ? Sketch a plot of this function, assuming the above assumptions.

(b) What is the Gibbs free energy ($G = H - TS$) difference, $\Delta G = G^{(u)} - G^{(f)}$, between the unfolded and folded states as a function of T ?

(c) What conditions on the parameters mean that this protein will unfold also at some low temperatures, given all the above assumptions? Sketch ΔG vs. T for such a case with two folding transitions.