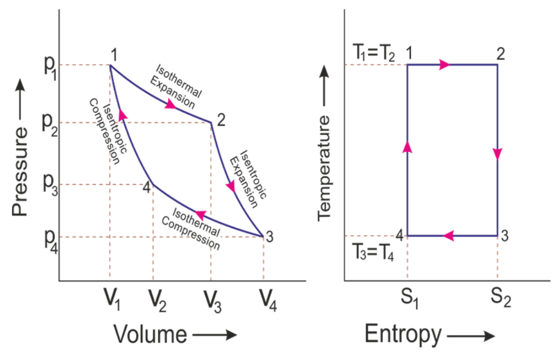
A Carnot engine uses n moles of ideal gas. A scientist wants to calculate T_C and T_H, but she does not have a thermometer. She does know all the volumes and the net work from one cycle. We wish to derive expressions for temperature.



(a) p-v diagram (b) T-S diagram

To find two unknowns we need two constraints. One is the work integral

$$W = \oint p dV$$

the rest come from adiabatic relations

$$p_2 V_2^\gamma = p_3 V_3^\gamma$$
$$p_4 V_4^\gamma = p_1 V_1^\gamma$$

Let's start with the integral. Using isothermal relation $pV = nRT$

$$\int_1^2 p dV = nRT_H \ln \left(\frac{V_2}{V_1} \right)$$

and

$$\int_3^4 p dV = -nRT_C \ln \left(\frac{V_3}{V_4} \right)$$

On the adiabatic legs $pV^\gamma = C$

$$\int_2^3 p dV = \frac{C}{\gamma - 1} \left(\frac{1}{V_2^{\gamma-1}} - \frac{1}{V_3^{\gamma-1}} \right)$$

but at either end point we may expand $C = pV^\gamma = (nRT)V^{\gamma-1}$. Thus

$$\int_2^3 p dV = \frac{nRT_C}{\gamma - 1} \left[\left(\frac{V_3}{V_2} \right)^{\gamma-1} - 1 \right]$$

and similarly

$$\int_4^1 p dV = -\frac{nRT_C}{\gamma - 1} \left[\left(\frac{V_4}{V_1} \right)^{\gamma-1} - 1 \right]$$

But we can connect the temperatures with the adiabatic relation

$$\begin{aligned} p_2 V_2^{\gamma} &= p_3 V_3^{\gamma} \\ nRT_H V_2^{\gamma-1} &= nRT_C V_3^{\gamma-1} \\ \frac{T_H}{T_C} &= \left(\frac{V_3}{V_2}\right)^{\gamma-1} \end{aligned}$$

Likewise

$$\frac{T_H}{T_C} = \left(\frac{V_4}{V_1}\right)^{\gamma-1}$$

Therefore

$$\frac{V_4}{V_1} = \frac{V_3}{V_2}$$

As a result the adiabatic integrals are equal in magnitude! [1]

$$\int_2^3 pdV + \int_4^1 pdV = 0$$

In addition to canceling the adiabatic legs, we can further simplify the work integral by observing $V_2/V_1 = V_3/V_4$. It follows that

$$W = \oint pdV = nR\ln\left(\frac{V_2}{V_1}\right)[T_H - T_C]$$

Using either of the adiabatic relations one can find

$$T_H = \frac{W}{nR\ln(V_2/V_1)}\left(\frac{V_3^{\gamma-1}}{V_3^{\gamma-1} - V_2^{\gamma-1}}\right)$$

$$T_C = \frac{W}{nR\ln(V_2/V_1)}\left(\frac{V_2^{\gamma-1}}{V_3^{\gamma-1} - V_2^{\gamma-1}}\right)$$

[1] Work is done, but without change in entropy ($\Delta S = 0$) the process is reversible. So an equal amount of energy is added and removed. Thus $\Delta W_{adiabatic} = 0$.