2 Jan 2022

Section B. Statistical Mechanics and Thermodynamics

1. **String Thermodynamics**

   An elastic string is found to have the following properties:

   - To stretch it to a total length x requires a force $f = \mu x - \alpha T + \beta T x$. Assume that α, β, μ are constants.
   - Its heat capacity at constant length x is proportional to temperature: $C(x) = A(x)T$.

   We can use thermodynamic identities to derive from these facts a variety of other thermal properties. More specifically:

   (a) Calculate $\frac{\partial S}{\partial x}|_T$.

   (b) Show that A has to be independent of x.

   (c) Calculate $\frac{\partial S}{\partial T}|_x$ and give the general expression for entropy $S(x, T)$ assuming $S(0, 0) = B$, where B is a constant.

   (d) Compute the heat capacity at zero tension $C_F = T\frac{\partial S}{\partial T}|_{f=0}$.

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a) Constant $T \to F = U - ST$

$$dF = -SdT - PdV \leftarrow P = -\frac{f}{A},\ dV = Adx,\ PdV = -fdx$$
$$= -SdT + fdx$$

$$\left.\begin{aligned}\left.\frac{dF}{dT}\right|_x = -S &\to \frac{d}{dx}\left(\left.\frac{dF}{dT}\right|_x\right)\Big|_T = -\frac{dS}{dx}\\ \left.\frac{dF}{dx}\right|_T = f &\to \frac{d}{dT}\left(\left.\frac{dF}{dx}\right|_T\right)\Big|_x = \frac{df}{dT}\end{aligned}\right\}\ \frac{d}{dx}\left(\left.\frac{dF}{dT}\right|_x\right)\Big|_T = \frac{d}{dT}\left(\left.\frac{dF}{dx}\right|_T\right)\Big|_x \to -\frac{dS}{dx} = \frac{df}{dT}$$

$$\frac{dS}{dx} = -\frac{df}{dT} = -\frac{d}{dT}(\mu x - \alpha T + \beta T x) = -(-\alpha + \beta x) = \alpha - \beta x$$

$$\boxed{\frac{\partial S}{\partial x} = \alpha - \beta x}$$

b) $C = T\frac{\partial S}{\partial T}$

$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}\left(T\frac{\partial S}{\partial T}\right) = T\frac{\partial^2 S}{\partial x \partial T} = T\frac{\partial S}{\partial T \partial x} = T\frac{\partial}{\partial T}\left(\frac{\partial S}{\partial x}\right) = T\frac{\partial}{\partial T}(\alpha - \beta x) = 0$$
$$\frac{\partial C}{\partial x} = \frac{\partial}{\partial x}(A(x)T) = A(x)\frac{\partial T}{\partial x} + T\frac{\partial A(x)}{\partial x} = T\frac{\partial A(x)}{\partial x} \equiv 0$$

$$\boxed{\frac{\partial A(x)}{\partial x} = 0,\ A \equiv \text{Const.}}$$

c) $C = T\left.\frac{\partial S}{\partial T}\right|_x \equiv TA \leftarrow \left.\frac{\partial S}{\partial T}\right|_x = A$

$$dS = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial T}dT \to S = \int\frac{\partial S}{\partial x}dx + \int\frac{\partial S}{\partial T}dT + S(0,0) = \alpha x - \frac{1}{2}\beta x^2 + AT + B$$

$$\boxed{\left.\frac{\partial S}{\partial T}\right|_x = A,\ S = \alpha x - \frac{1}{2}\beta x^2 + AT + B}$$

d) $$C_F = T \frac{\partial S}{\partial T}\Big|_{f=0}$$

$$= T \frac{\partial}{\partial T}\left(\alpha x - \frac{1}{2}\beta x^2 + AT + B\right) = T\left(\alpha \frac{\partial x}{\partial T} - \beta x \frac{\partial x}{\partial T} + A\right)$$

$$f = \mu x - \alpha T + \beta T x = 0 \rightarrow df = \mu dx - \alpha dT + \beta T dx + \beta x dT = 0$$

$$(\mu + \beta T) dx = (\alpha - \beta x) dT \rightarrow \frac{dx}{dT} = \frac{\alpha - \beta x}{\mu + \beta T}$$

$$C_F = AT + T \frac{\partial x}{\partial T}(\alpha - \beta x) = AT + T \frac{(\alpha - \beta x)^2}{\mu + \beta T}$$

$$\boxed{C_F = AT + T \frac{(\alpha - \beta x)^2}{\mu + \beta T}}$$