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2. **Decay Angular Correlations**

An unpolarized nucleus of spin S (to be determined) decays into a nucleus of spin 0, plus two alpha particles. The alpha particles have spin 0 and there are of course many possibilities for their orbital angular momentum. Let us consider the case that both have orbital angular momentum $L = 1$. By angular momentum addition, the original nucleus could have had $S = 2, 1, \text{or } 0$. We can distinguish the three cases by measuring the probability distribution of the angle between the directions of motion of the outgoing alphas. Since the original nucleus is unpolarized, there is no other meaningful angle in the problem.

(a) As a first step, use the techniques of angular momentum addition to construct states of total angular momentum 2 out of two particles of orbital angular momentum 1; that is, find the linear combinations of $Y_{1m_1}(\theta_1, \phi_1)Y_{1m_2}(\theta_2, \phi_2)$ that transform in the angular momentum 2 representation.

(b) Next, compute the probability distribution of the angle between the two alphas in the case that the original $S = 2$ nucleus is unpolarized (*i.e.* has equal *probability* of being in the 5 different S_z substates). Work is simplified, at no cost in generality, by assuming that both alphas lie in the plane perpendicular to the quantization axis (so that in your spherical harmonics $\theta = \pi/2$ and only ϕ varies).

(c) Next, do the same computation for the case that the intial nucleus has $S = 0$ and compare with b) to show that the two cases can be distinguished.

(d) Why do we not ask you to consider the case $S = 1$?

You will need the $L = 1$ spherical harmonics:

$$Y_{10}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos\theta \qquad Y_{1\pm1}(\theta, \phi) = \sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

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a) 1×1 states of $s=2$

$$\boxed{|2,2\rangle = Y_{11}Y_{11}}$$

$$\hat{L}_-|2,2\rangle = \hbar\sqrt{2(2+1)-2(2-1)}\,|2,1\rangle = \hbar\sqrt{6-2}\,|2,1\rangle = 2\hbar|2,1\rangle$$

$$\hat{L}_- Y_{11}Y_{11} = \hbar\sqrt{1(1+1)-1(1-1)}\left(Y_{10}Y_{11}+Y_{11}Y_{10}\right)$$

$$= \sqrt{2}\,\hbar\left(Y_{10}Y_{11}+Y_{11}Y_{10}\right)$$

$$\boxed{|2,1\rangle = \frac{1}{\sqrt{2}}\left(Y_{10}Y_{11}+Y_{11}Y_{10}\right)}$$

$$\hat{L}_-|2,1\rangle = \hbar\sqrt{2(2+1)-1(1-1)}\,|2,0\rangle = \sqrt{6}\,\hbar|2,0\rangle$$

$$\hat{L}_-\left(Y_{10}Y_{11}+Y_{11}Y_{10}\right) = \left(Y_{1-1}Y_{11}\right)\left(\hbar\sqrt{1(1+1)-0(0-1)}\right)$$
$$+2\left(Y_{10}Y_{10}\right)\left(\hbar\sqrt{1(1+1)-1(1-1)}\right)$$
$$+\left(Y_{11}Y_{1-1}\right)\left(\hbar\sqrt{1(1+1)-0(0-1)}\right)$$
$$= \hbar\left[\sqrt{2}\,Y_{1-1}Y_{11} + 2\sqrt{2}\,Y_{10}Y_{10} + \sqrt{2}\,Y_{11}Y_{1-1}\right]$$

$$\boxed{|2,0\rangle = \frac{1}{\sqrt{6}}\left(Y_{1-1}Y_{11} + 2Y_{10}Y_{10} + Y_{11}Y_{1-1}\right)}$$

$$\boxed{|2,-2\rangle = Y_{1-1}Y_{1-1}}$$

$$\hat{L}_+|2,-2\rangle = \hbar\sqrt{2(2+1)+2(-2+1)}\,|2,-1\rangle = 2\hbar|2,-1\rangle$$

$$\hat{L}_+ Y_{1-1}Y_{1-1} = \hbar\sqrt{1(1+1)+1(-1+1)}\left(Y_{10}Y_{1-1}+Y_{1-1}Y_{10}\right) = \sqrt{2}\,\hbar\left(Y_{10}Y_{1-1}+Y_{1-1}Y_{10}\right)$$

$$\boxed{|2,-1\rangle = \frac{1}{\sqrt{2}}\left(Y_{10}Y_{1-1}+Y_{1-1}Y_{10}\right)}$$

b) $Y_{10}(\phi)=0$, $Y_{1\pm1}(\phi)=\sqrt{\frac{3}{8\pi}}e^{\pm i\phi}$

$$\psi_{2,2}=Y_{11}Y_{11}=\frac{3}{8\pi}e^{2i\phi}$$
$$\psi_{2,1}=\frac{1}{\sqrt{2}}(Y_{10}Y_{11}+Y_{11}Y_{10})=0$$
$$\psi_{2,0}=\frac{1}{\sqrt{6}}(Y_{1-1}Y_{11}+2Y_{10}Y_{10}+Y_{11}Y_{1-1})=\frac{1}{\sqrt{6}}\left(\frac{3}{8\pi}e^{-i(\phi_1-\phi_2)}+\frac{3}{8\pi}e^{i(\phi_1-\phi_2)}\right)$$
$$=2\frac{1}{\sqrt{6}}\frac{3}{8\pi}\cos(\phi_1-\phi_2)=\frac{1}{\sqrt{6}}\sqrt{\frac{9}{16\pi^2}}\cos(\phi_1-\phi_2)=\sqrt{\frac{3}{32\pi^2}}\cos(\phi_1-\phi_2)$$
$$\psi_{2,-1}=\frac{1}{\sqrt{2}}(Y_{10}Y_{1-1}+Y_{1-1}Y_{10})=0$$
$$\psi_{2,-2}=Y_{1-1}Y_{1-1}=\frac{3}{8\pi}e^{-2i\phi}$$

$$P=|\psi|^2$$
$$P_{2,2}=\left(\frac{3}{8\pi}\right)^2,\ P_{2,1}=0,\ P_{2,0}=\frac{3}{32\pi^2}\cos^2(\phi_1-\phi_2),\ P_{2,-1}=0,\ P_{2,-2}=\left(\frac{3}{8\pi}\right)^2$$

$$P_2(\Delta)=\sum_{m=2}^{2}P_{2,m}=\left(\frac{3}{8\pi}\right)^2\left(2+\frac{2}{3}\cos^2(\phi_1-\phi_2)\right)=\boxed{\left(\frac{3}{8\pi}\right)^2\left(2+\frac{2}{3}\cos^2(\Delta)\right)}$$

c) $|1,1\rangle=\frac{1}{\sqrt{2}}(Y_{11}Y_{10}-Y_{10}Y_{11})\leftarrow$ ($|2,1\rangle$ but swap coeffs. and make one negative)

$|1,-1\rangle=\frac{1}{\sqrt{2}}(Y_{1-1}Y_{10}-Y_{10}Y_{1-1})$

$$\hat{L}_-|1,1\rangle=\hbar\sqrt{1(1+1)-1(1-1)}\,|1,0\rangle=\sqrt{2}\hbar|1,0\rangle$$
$$\hat{L}_-(Y_{11}Y_{10}-Y_{10}Y_{11})=-\hbar\sqrt{1(1+1)-0(0-1)}\,Y_{1-1}Y_{11}+\hbar\sqrt{1(1+1)-0(0-1)}\,Y_{11}Y_{1-1}$$
$$-\hbar\sqrt{1(1+1)-1(1-0)}\,Y_{10}Y_{10}+\hbar\sqrt{1(1+1)-1(1-0)}\,Y_{10}Y_{10}$$
$$=\sqrt{2}\hbar(Y_{11}Y_{1-1}-Y_{1-1}Y_{11})$$
$$|1,0\rangle=\frac{1}{\sqrt{2}}(Y_{11}Y_{1-1}-Y_{1-1}Y_{11})$$

$$|0,0\rangle=aY_{1-1}Y_{11}+bY_{10}Y_{10}+cY_{11}Y_{1-1}$$
$$\langle 2,0|0,0\rangle=0,\ \langle 1,0|0,0\rangle=0,\ \langle 0,0|0,0\rangle=|a|^2+|b|^2+|c|^2=1$$

$$\langle 1,0|0,0\rangle = \frac{1}{\sqrt{2}}a - \frac{1}{\sqrt{2}}c = 0 \Leftarrow a = c$$

$$\langle 2,0|0,0\rangle = \frac{1}{\sqrt{6}}(a + 2b + c) = 0 \Leftarrow a = -b = c$$

$$\langle 0,0|0,0\rangle = a^2 + a^2 + a^2 = 1 = 3a^2;\ a = \frac{1}{\sqrt{3}} = c,\ b = -\frac{1}{\sqrt{3}}$$

$$|0,0\rangle = \frac{1}{\sqrt{3}}\left(Y_{1-1}Y_{11} - Y_{10}Y_{10} + Y_{11}Y_{1-1}\right)$$

$$\psi_{0,0} = \frac{1}{\sqrt{3}}\left(\frac{3}{8\pi}e^{-i(\phi_1-\phi_2)} + \frac{3}{8\pi}e^{i(\phi_1-\phi_2)}\right)$$

$$= \frac{1}{\sqrt{3}}\sqrt{\frac{9}{64\pi^2}}\,2\cos(\Delta)$$

$$= \sqrt{\frac{3}{16\pi^2}}\cos(\Delta)$$

$$P_{0,0} = |\psi|^2 = \frac{3}{16\pi^2}\cos^2(\Delta)$$

$$P_0(\Delta) = P_{0,0} = \boxed{\frac{3}{16\pi^2}\cos^2(\Delta)}$$

$\boxed{P_2 \propto 2 + \frac{2}{3}\cos^2(\Delta) \text{ and } P_0 \propto \cos^2(\Delta) \Leftarrow \text{can distinguish}}$

d) Around $\Delta \approx 0$, P_2 and P_0 look similar close to $\Delta \approx 0$

If P_1 looks different that P_2 and P_0 at $\Delta = 0$, not worth investigating

$$\psi_{1,1} = \psi_{1,-1} = 0,\quad \psi_{1,0} = \sqrt{\frac{3}{16\pi^2}}\sin(\Delta) \rightarrow P_1 = \frac{3}{16\pi^2}\sin^2(\Delta)$$

$\boxed{P_1 \propto \sin^2(\Delta), \text{ which is easily distinguisible from } P_2 \text{ and } P_0}$