Section A. Quantum Mechanics

1. Perturbed Harmonic Oscillator (MR solution)

A particle of mass \( m \) moves one-dimensionally in a static harmonic oscillator potential

\[
V = \frac{1}{2} m \omega^2 x^2
\]

It is also acted on by a space-time dependent perturbation potential \( W(x, t) \) that is narrowly localized around a point \( x_0(t) \) in space that moves with time. To simulate this, take the delta function expression

\[
W = \lambda \delta(x - x_0(t))
\]

where \( \lambda \) parametrizes the potential strength.

Let \( x_0(t) = vt \) for some velocity \( v \) and suppose that the particle was in the oscillator ground state \( u_0(x) \) in the remote past (at time \( t \to -\infty \)). What is the probability that the particle will be found in the first excited oscillator state \( u_1(x) \) in the remote future? Treat \( W \) as a small perturbation and work out the answer to lowest order in \( \lambda \). Sketch the dependence of the transition probability on \( v \) and identify the value of \( v \) that maximizes the transition probability.

You are reminded that

\[
\begin{align*}
  u_0 &= \left( \frac{1}{\pi a^2} \right)^{1/4} \exp\left(-\frac{x^2}{2a^2}\right), &
  u_1 &= \sqrt{2} \frac{x}{a} u_0, &
  a &= \sqrt{\frac{\hbar}{m \omega}}.
\end{align*}
\]
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Trust as two-level system \( \Rightarrow \) time-dependent perturbation theory

\[ C_2 = -\frac{i}{\hbar} \int_{-\infty}^{\infty} \langle \gamma_2 | W(t) | \gamma_0 \rangle e^{\frac{-i E_0 t}{\Delta}} dt \leq P_2 = |c_0|^2 \]

\[ E_1 - E_0 = \hbar \omega \]

\[ \langle \gamma_2 | W(t) | \gamma_0 \rangle = \int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{\pi}} U_0^\dagger(x) \phi S(x-Vt) dx \]

\[ = \sqrt{\frac{\pi}{\lambda}} \frac{Vt}{\alpha} U_0^\dagger(Vt) = U_0^\dagger(x) = \frac{\lambda}{\sqrt{\pi}} \exp \left( -\frac{x^2}{\alpha^2} \right) \]

\[ = \sqrt{\frac{\pi}{\lambda}} \frac{2Vt}{\alpha^2} \exp \left( -\frac{V^2 t^2}{\alpha^2} \right) \]

\[ = -\frac{i}{\hbar} \sqrt{\frac{\lambda}{\pi}} \frac{2V}{\alpha^2} \int_{-\infty}^{\infty} U \exp \left[ -\left( \frac{V^2 t^2}{\alpha^2} \right) \right] dt' \leq U = \frac{Vt}{\alpha}, t' = \frac{u}{V}, \alpha = \frac{\alpha}{V} \]

\[ = -\frac{i}{\hbar} \sqrt{\frac{\lambda}{\pi}} \frac{2V}{\alpha^2} \int_{-\infty}^{\infty} U \exp \left[ -\left( \frac{u^2}{\alpha^2} \right) \right] du \]

\[ = -\frac{i}{\hbar} \sqrt{\frac{\lambda}{\pi}} \frac{V}{\alpha} \int_{-\infty}^{\infty} U \exp \left[ -\left( \frac{i\omega u}{\alpha} \right)^2 \right] \exp \left[ -\frac{\omega^2 a^2}{V^2} \right] du \leq a = \frac{1}{\sqrt{n}} \]

\[ = -\frac{i}{\hbar} \sqrt{\frac{\lambda}{\pi}} \frac{V}{\alpha} \exp \left( -\frac{\omega^2 a^2}{V^2} \right) \int_{-\infty}^{\infty} \exp \left[ -\frac{\omega^2 a^2}{V^2} \right] \exp \left[ -\frac{i\omega u}{\alpha} \right] \exp \left[ -\frac{\omega^2 a^2}{V^2} \right] du \]

\[ = -\frac{i}{\hbar} \sqrt{\frac{\lambda}{\pi}} \frac{V}{\alpha} \exp \left( -\frac{\omega^2 a^2}{V^2} \right) \int_{-\infty}^{\infty} \exp \left[ -\frac{i\omega u}{\alpha} \right] \exp \left[ -\frac{\omega^2 a^2}{V^2} \right] \exp \left[ -\frac{i\omega u}{\alpha} \right] \exp \left[ -\frac{\omega^2 a^2}{V^2} \right] du \]

\[ = -\frac{i}{\hbar} \sqrt{\frac{\lambda}{\pi}} \frac{V}{\alpha} \exp \left( -\frac{\omega^2 a^2}{V^2} \right) \leq a = \frac{1}{\sqrt{n}} \]

\[ = \frac{\lambda}{\sqrt{\pi} \hbar \omega} \exp \left( -\frac{\omega^2 a^2}{V^2} \right) \]

\[ P_2 = |c_0|^2 = \frac{1}{V^4} \frac{m \hbar^2}{\lambda \omega} \exp \left( -\frac{1}{V^2} \frac{\omega \lambda}{2\pi} \right) \leq V_0^4 = \frac{\omega^2}{2\pi}, V_0^2 = \frac{\hbar \omega}{2\pi} \]

\[ = \left( \frac{\lambda}{V^4} \right) \exp \left( -\frac{\lambda^2}{V^2} \right) \]

\[ P_2 = \left( \frac{V_0}{V} \right)^4 \exp \left[ -\left( \frac{V_0^2}{V} \right)^2 \right], V_0^4 = \frac{\omega^2}{2\pi}, V_0^2 = \frac{\hbar \omega}{2\pi} \]
Max of Probability at \( \frac{dP_z}{dV} = 0 \)

\[
\frac{dP_z}{dV} = -4 \frac{V_0^4}{V^3} \exp\left(-\left(\frac{V_0}{V}\right)^2\right) + \left(\frac{V_0}{V}\right)^4 \exp\left(-\left(\frac{V_0}{V}\right)^2\right)(-1)(-2)\frac{V^2}{V^3} = 0
\]

\[
\frac{4}{V^3} = 2 \frac{V_0^2}{V^2} \rightarrow V^2 = \frac{1}{2} V_0^2 = \frac{2u}{m}
\]

\[
V_{max} = \sqrt{\frac{2u}{m}}
\]