PROBLEM J16Q.3

(a) Using the suggested gauge, we have $A_x = A_z = 0$ everywhere with

$$A_y = \begin{cases} 0, & x < 0, \\ B_0 x c, & 0 < x < d, \\ B_0 d c, & x > d. \end{cases}$$

Then the Schrödinger equation becomes

$$\nabla^2 \Psi - \frac{ie}{\hbar c} A_y(x)\, \partial_y \Psi - \frac{e^2}{\hbar^2 c^2} A_y(x)^2 \Psi + k^2 \Psi = 0.$$

For an incident wave e^{ikx}, by translational symmetry along y, z the entire scattered solution must depend only on x. Thus we may reduce to the effective equation of motion

$$\partial_x^2 \Psi(x) + \left(k^2 - \frac{e^2}{\hbar^2 c^2} A_y(x)^2 \right) \Psi(x) = 0.$$

For $x < 0$, we have $A_y(x) = 0$ and so the incident and reflected waves e^{ikx}, Re^{-ikx} indeed satisfy the reduced equation. For $x > d$, the equation of motion becomes

$$\partial_x^2 + \left(k^2 - \frac{e^2 B_0^2 d^2}{\hbar^2} \right) \Psi(x) = 0,$$

which has the solution $\exp\left(i\hat{k}x\right)$ with transmitted wavevector

$$\hat{k} := \sqrt{k^2 - \left(B_0 d \frac{e}{\hbar}\right)^2}.$$

(b) A classical particle of kinetic energy E would have velocity $v = \sqrt{2E/m}$, and follows a circular trajectory of radius $r = mv/eB_0$ within the strip. If $r < d$, then classically the particle cannot travel through the strip, which occurs for the critical energy

$$E_0 = \boxed{\frac{(eB_0 d)^2}{2m}.}$$

In the quantum description, the transmitted wavevector $\hat{k}$ becomes imaginary when

$$k = eB_0 d/\hbar.$$

This corresponds to an incident energy

$$E = \frac{\hbar^2 k^2}{2m} = \frac{(eB_0 d)^2}{2m},$$

which is equal to E_0 as expected.

(c) The transmitted probability flux (for $x > d$) is given by

$$\frac{\hbar}{2mi} \left(\Psi^* \nabla \Psi - \frac{ie}{\hbar c} A_y(x)\, \Psi^* \Psi\, \hat{\mathbf{y}} \right) + \text{h.c.} = \frac{\hbar}{2mi} |T|^2 \left(i\hat{k}\, \hat{\mathbf{x}} - \frac{ie}{\hbar c} B_0 d c\, \hat{\mathbf{y}} \right) + \text{h.c.}$$
$$= \boxed{\frac{|T|^2}{m} \left(\hbar \hat{k}\, \hat{\mathbf{x}} - eB_0 d\, \hat{\mathbf{y}} \right)},$$

which does not lie purely along the x direction.

Classically, within the strip $0 < x < d$ the particle is deflected downwards by the magnetic field along a circular arc, and exits with some negative y momentum. The angle θ subtended by this arc will satisfy

$$d = r\sin\theta, \quad \text{where} \quad r = \frac{mv}{eB_0}.$$

Thus the vertical component of the momentum after transmission is

$$-mv\sin\theta = -\frac{mvd}{r} = -eB_0 d.$$

This is precisely the component $-eB_0 d\,\hat{\mathbf{y}}$ present in the quantum probability flux.

(d) In this limit, we simply need to impose differentiability of the wavefunction across $x = 0$. This condition yields

$$T = R + 1 \quad \text{and} \quad i\hat{k}T = -ikR + ik,$$

which has the solution

$$T = \frac{2}{1 + \hat{k}/k} \quad \text{and} \quad R = \frac{1 - \hat{k}/k}{1 + \hat{k}/k}, \quad \text{where} \quad \hat{k} = \sqrt{k^2 - (eB_0 d/\hbar)^2}.$$

These coefficients are reminiscent of the corresponding coefficients from classical optics, if we identify $\hat{k}/k$ with the refractive index.