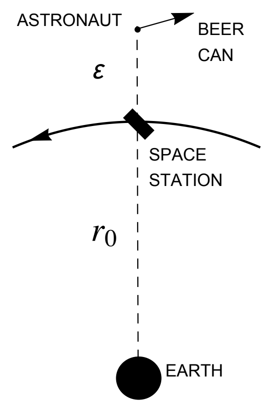
21 Dec 2021

3. **Orbiting Beer Can**

A space station is in a circular orbit about the earth at a radius r_0. An astronaut on a space walk happens to be distance ε from the station on the line joining the station to the center of the earth. With practice the astronaut can throw a beer can so that it appears to orbit the space station, in the plane of the space station's orbit about the earth. We can neglect the gravitational attraction between the beer can and the space station, so the beer can is just orbiting the Earth in a slightly non-circular orbit, cleverly chosen so that the beer can never drifts away from the space station as the latter moves in its own circular orbit. Let's work out the details of this trick.



(a) First a helpful lemma: Use the radial equation for orbits of a given specific angular momentum $\hat{\ell} = \ell/m$ to show that the period of *small* radial oscillations about the circular orbit is the same as the period of the circular orbit itself.

(b) The astronaut launches the beer can with zero radial component of velocity, and with tangential velocity chosen so that its specific angular momentum $\hat{\ell} = \ell/m$ equals that of the space station. Find the subsequent motion of the beer can in Earth-centered radial coordinates $(r(t), \theta(t))$, working to the first approximation in small deviations from a circular orbit.

(c) Construct the orbit of the beer can relative to the space station, using x-y coordinate axes centered on the station, rotating with the station, so that the negative y-axis always points down toward the center of the Earth. Show that the orbit in these station-centric coordiantes is an ellipse and give its period, semi-major and semi-minor axes.

J16M.3

a) hold angular momentum costant; $l = mr^2\dot{\theta} \to \hat{l} = r^2\dot{\theta}$

$$V_{eff} = \frac{1}{2} m r^{-2} \hat{l}^2 - GMm r^{-1}$$

$$L = \frac{1}{2} m \dot{r}^2 - \frac{1}{2} m r^{-2} \hat{l}^2 + GMm r^{-1}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} \; ; \; \frac{\partial L}{\partial r} = m r^{-3} \hat{l}^2 - GMm r^{-2}$$

$$\ddot{r} = r^{-3}\hat{l}^2 - r^{-2} GM$$

Circular: $\ddot{r} = 0, r = r_0 \to \hat{l}_0 = \sqrt{GMr_0}$

perturbed circular orbit: $r = r_0 + \varepsilon$, $\ddot{r} = \ddot{\varepsilon}$

$$\ddot{\varepsilon} = \hat{l}^2 (r_0 + \varepsilon)^{-3} - GM(r_0+\varepsilon)^{-2} = r_0^{-3}\hat{l}^2\left(1 - 3\frac{\varepsilon}{r_0}\right) - r_0^{-2} GM\left(1 - 2\frac{\varepsilon}{r_0}\right)$$

$$= (r_0^{-3}\hat{l}^2 - r_0^{-2}GM) - (3 r_0^{-4}\hat{l}^2 - 2 r_0^{-3} GM)\varepsilon$$

$$= -(3 r_0^{-4}\hat{l}^2 - 2 r_0^{-3}GM)\varepsilon = -\omega_{osc}^2 \varepsilon$$

$$\boxed{\omega_{osc} = \sqrt{3\frac{\hat{l}^2}{r_0^4} - 2\frac{GM}{r_0^3}}}$$

$$\omega_{osc} = (3 r_0^{-4}\hat{l}^2 - 2 r_0^{-3}GM)^{1/2} \leftarrow \hat{l}^2 = GMr_0$$

$$= (3 r_0^{-3}GM - 2 r_0^{-3}GM) = \sqrt{GM/r_0^3} = \omega_{orb}$$

$$\boxed{\omega_{osc} = \sqrt{\frac{GM}{r_0^3}} = \omega_{orb}}$$

For future: $\omega_{orb} = \omega_{osc} = \sqrt{GM/r_0^3} \equiv \omega$

b) $\ddot{\xi} = -\omega^2 \xi \to \xi(t) = A\cos(\omega t) + B\sin(\omega t) \leftarrow \xi(0) = \varepsilon,\ \dot{\xi}(0) = 0$

$\xi(t) = \varepsilon \cos(\omega t) \longleftarrow r(t) = r_0 + \xi(t)$

$\dot{\theta}(t) = \frac{\hat{l}}{r^2} = r_0^{-2}\hat{l}\left(1 + \frac{\varepsilon}{r_0}\cos(\omega t)\right)^{-2} \simeq r_0^{-2}\hat{l}\left(1 - 2\frac{\varepsilon}{r_0}\cos(\omega t)\right)$

$\theta(t) = \int_0^t \dot{\theta}(t')dt' = r_0^{-2}\hat{l}\int_0^t 1 - 2\frac{\varepsilon}{r_0}\cos(\omega t')\,dt' = r_0^{-2}\hat{l}\left(t - 2\frac{\varepsilon}{r_0\omega}\sin(\omega t)\right)$

$$\boxed{\begin{gathered} r(t) = r_0 + \varepsilon\cos(\omega t) \\ \theta(t) = \frac{\hat{l}}{r_0^2}\left(t - 2\frac{\varepsilon}{r_0\omega}\sin(\omega t)\right) \end{gathered}}$$

c) $y(t) = r(t) - r_0 \leftarrow \hat{y}|_{station} = \hat{r}|_{earth}$

$= \varepsilon\cos(\omega t)$

$x(t) = r_0(\theta(t) - \theta_{station}(t)) \leftarrow \hat{x}|_{station} = \hat{\theta}_{earth},\ \theta_{station} = \omega t = \frac{\hat{l}}{r_0^2}t$

$= -2\frac{\hat{l}\varepsilon}{r_0^2\omega}\sin(\omega t) \leftarrow \hat{l} = r_0^2\omega$

$= -2\varepsilon\sin(\omega t)$

$$\boxed{\begin{gathered} \text{Period}: \frac{2\pi}{\omega} \\ \text{Semimajor}: 2\varepsilon \\ \text{Semiminor}: \varepsilon \end{gathered}}$$