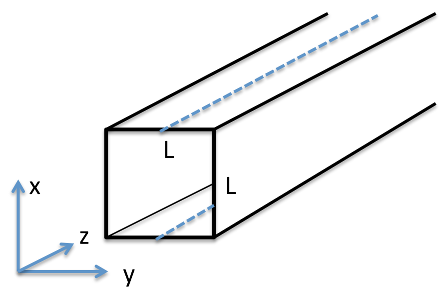
21 Dec 2021

3. **Slicing a Waveguide**

A square waveguide with perfectly conducting walls at $x = 0$, $x = L$, $y = 0$, and $y = L$ extends along the z-axis. A waveguide like this has many modes that propagate in the z-direction. Consider the restricted set of propagating modes for which the E-field has only an x-component:

$$\vec{E} = (E_x, 0, 0) \qquad E_x = E_0 \sin(m\pi y/L)\sin(kz - \omega t) \quad m = 1, 2, \ldots$$

and the B-field only has components in the orthogonal directions, $\vec{B} = (0, B_y, B_z)$.



(a) Show that the specified form of the E-field satisfies the vacuum Maxwell equation $\vec{\nabla} \cdot \vec{E} = 0$ inside the wave guide, and also satisfies the appropriate boundary condition at the conducting walls.

(b) Write down expressions for the $B_{y,z}(y, z, t)$ components that must accompany this E-field in order to satisfy the vacuum Maxwell equation $-\partial\vec{B}/\partial t = \vec{\nabla} \times \vec{E}$. Show that they satisfy the appropriate boundary conditions at the conducting walls.

(c) For each of these modes, find the dispersion relation $\omega(k)$ that guarantees that all the vacuum Maxwell equations are satisfied.

(d) The cavity is now sawed in half on the plane $y = L/2$, but not pulled apart (along the dotted lines in the figure). This means that no surface currents can flow across the cut line. What are the allowed mode frequencies now? What is the answer to this question if the cut is made along $x = L/2$?

J16E.3

a) $\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\partial}{\partial x}\left(E_0 \sin\left(\frac{m\pi}{L}y\right)\sin(kz-wt)\right) = 0$

Boundary at walls: $\vec{E}^{\parallel} = \vec{0}$

At yz-wall at $x=0$: $(E_y, E_z) = (0,0)$

At yz-wall at $x=L$: $(E_y, E_z) = (0,0)$

At xz-wall at $y=0$: $(E_x, E_z) = (E_0\sin(0)\sin(kz-wt), 0) = \vec{0}$

At xz-wall at $y=L$: $(E_x, E_z) = (E_0\sin(m\pi)\sin(kz-wt), 0) = \vec{0}$

$$\boxed{\vec{\nabla}\cdot\vec{E} \text{ and } \vec{E}^{\parallel}_{walls} = \vec{0}}$$

b)

$$-\vec{\nabla}\times\vec{E} = -\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\left[-\left(0-\frac{\partial E_x}{\partial z}\right)\hat{y} + \left(0-\frac{\partial E_x}{\partial y}\right)\hat{z}\right] = -\frac{\partial E_x}{\partial z}\hat{y} + \frac{\partial E_x}{\partial y}\hat{z}$$

$$\vec{B} = -\vec{\nabla}\times\left[\int \vec{E}\,dt\right] = -\frac{\partial}{\partial z}\left(\int E_x dt\right)\hat{y} + \frac{\partial}{\partial y}\left(\int E_x dt\right)\hat{z}$$

$\int E_x dt \equiv \mathcal{E}_x = \frac{E_0}{\omega}\sin\left(\frac{m\pi}{L}y\right)\cos(kz-wt) = \mathcal{E}_0\sin\left(\frac{m\pi}{L}y\right)\cos(kz-wt) \leftarrow \mathcal{E}_0 \equiv \frac{E_0}{\omega}$

$\frac{\partial \mathcal{E}_x}{\partial y} = \mathcal{E}_0\left(\frac{m\pi}{L}\cos\left(\frac{m\pi}{L}y\right)\cos(kz-wt)\right)$

$\frac{\partial \mathcal{E}_x}{\partial z} = \mathcal{E}_0\left(-k\sin\left(\frac{m\pi}{L}y\right)\sin(kz-wt)\right)$

$= \mathcal{E}_0\left(k\sin\left(\frac{m\pi}{L}y\right)\sin(kz-wt)\hat{y} + \frac{m\pi}{L}\cos\left(\frac{m\pi}{L}y\right)\cos(kz-wt)\hat{z}\right)$

$$\boxed{\vec{B} = \frac{E_0}{\omega}\left(k\sin\left(\frac{m\pi}{L}y\right)\sin(kz-wt)\hat{y} + \frac{m\pi}{L}\cos\left(\frac{m\pi}{L}y\right)\cos(kz-wt)\hat{z}\right)}$$

At walls, $B^{\perp} = 0$

At yz-wall at $x=0$: $B_x = 0$

At yz-wall at $x=L$: $B_x = 0$

At xz-wall at $y=0$: $B_y = \varepsilon_0 K \sin(0)\sin(kz-\omega t) = 0$

At xz-wall at $y=L$: $B_y = \varepsilon_0 K \sin(m\pi)\sin(kz-\omega t) = 0$

$$\boxed{B^{\perp}_{walls} = 0}$$

c) Wave Eqn: $\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \to \nabla^2 E_x = \frac{1}{c^2}\frac{\partial E_x}{\partial t}$

$\nabla^2 E_x = -\left(\frac{m\pi}{L}\right)^2 E_x - k^2 E_x$, $\frac{\partial E_x}{\partial t} = -\omega^2 E_x$

$\frac{\omega^2}{c^2} = \left(\frac{m\pi}{L}\right)^2 + k^2$

$\omega^2 = c^2\left[\left(\frac{m\pi}{L}\right)^2 + k^2\right]$

$$\boxed{\omega(k) = c\sqrt{k^2 + \left(\frac{m\pi}{L}\right)^2}}$$

d) No currents can pass through slice: $B_y = 0$ at $y = \frac{L}{2}$, for first case

$B_y(y=\frac{L}{2}) = \varepsilon_0 K \sin\left(\frac{m\pi}{2}\right)\sin(kz-\omega t) = 0 \to \sin\left(\frac{m\pi}{2}\right) = 0 \to \frac{m\pi}{2} = n\pi \leftarrow n \in \mathbb{Z}$

$\therefore m = 2n$ or $m' = 2m$

$$\boxed{\omega(k) = c\sqrt{k^2 + \left(\frac{2m\pi}{L}\right)^2}}$$

For second case, $B_x = 0$ at $x = L/2$

$B_x = 0$ everywhere already, and thus causes no change