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2. **Long Antenna Pattern**

A thin, straight, conducting wire centered on the origin and oriented along the z-axis carries a current

$$I = \hat{z} I_0 \cos \omega t$$

everywhere along its length ℓ. This antenna will radiate electromagnetic waves with frequency ω and wavelength $\lambda = 2\pi c/\omega$. We will not assume that $\ell << \lambda$.

(a) Because of the current, a time-dependent charge $q(t)$ will accumulate at the two ends of the wire. Give expressions for the charge and current *densities* $\rho(\vec{x}, t)$ and $\vec{j}(\vec{x}, t)$ on the wire. Show that the electric dipole moment of this charge distribution satisfies $p(t) = p_0 \sin(\omega t)$ and evaluate p_0

(b) Use these source densities to construct the scalar and vector potentials everywhere outside the source region ($r \gg \ell$). Do not assume anything about the relative magnitudes of ℓ and λ. Do state the gauge you are using.

(c) Compute the angular distribution of the energy flux radiated from this antenna. Show that it reduces to the standard electric dipole radiation pattern when $\lambda \gg \ell$. For general λ, show that the energy flux radiated perpendicular to the $\hat{z}$ direction depends only on the maximum electric dipole moment p_0 (and agrees with the standard electric dipole radiation result).

a) $\vec{I} = \hat{z} I_0 \cos(\omega t)$

$$\boxed{\vec{J} = \hat{z} I_0 \cos(\omega t)\,\delta(x)\delta(y)}$$

$$I(t) = \frac{dq}{dt} \to q(t) = \frac{I_0}{\omega}\sin(\omega t)$$

$$\boxed{\rho = q(t)\cdot\frac{1}{V} = \frac{I_0}{\omega}\sin(\omega t)\,\delta(x)\delta(y)\left(\delta\left(z-\frac{l}{2}\right) - \delta\left(z+\frac{l}{2}\right)\right)}$$

$$\boxed{p(t) = q(t)\cdot l = \frac{I_0 l}{\omega}\sin(\omega t) \to p_0 = \frac{I_0 l}{\omega}}$$

b) $$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\rho(\vec{r}', t - |\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{I_0}{\omega}\sin\left(\omega t - \frac{((x-x')^2+(y-y')^2+(z-z')^2)^{1/2}}{c/\omega}\right)\frac{\delta(x')\delta(y')\left(\delta(z'-\frac{l}{2})-\delta(z'-\frac{l}{2})\right)}{((x-x')^2+(y-y')^2+(z-z')^2)^{1/2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} dz \frac{I_0}{\omega}\sin\left(\omega t - \frac{(x^2+y^2+(z-z')^2)^{1/2}}{c/\omega}\right)\frac{\delta(z'-\frac{l}{2})}{(x^2+y^2+(z-z')^2)^{1/2}}$$

$$- \frac{1}{4\pi\varepsilon_0}\int_{-\infty}^{\infty} dz \frac{I_0}{\omega}\sin\left(\omega t - \frac{(x^2+y^2+(z-z')^2)^{1/2}}{c/\omega}\right)\frac{\delta(z'+\frac{l}{2})}{(x^2+y^2+(z-z')^2)^{1/2}}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{I_0}{\omega}\left[\sin\left(\omega t - \frac{(x^2+y^2+(z-\frac{l}{2})^2)^{1/2}}{c/\omega}\right)\frac{1}{(x^2+y^2+(z-\frac{l}{2})^2)^{1/2}} - \sin\left(\omega t - \frac{(x^2+y^2+(z+\frac{l}{2})^2)^{1/2}}{c/\omega}\right)\frac{1}{(x^2+y^2+(z+\frac{l}{2})^2)^{1/2}}\right]$$

$$r_\pm = \left(x^2+y^2+z^2 \pm zl - \frac{l^2}{4}\right)^{1/2} = r\left(1 \pm \frac{z}{r}\frac{l}{r} - \frac{1}{4}\frac{l^2}{r^2}\right)^{1/2} \leftarrow x^2+y^2+z^2 \equiv r^2$$

$$= r\left(1 \pm \frac{l}{r}\cos\theta - \frac{1}{4}\frac{l^2}{r^2}\right)^{1/2} \approx r\left(1 \pm \cos\theta\frac{a}{r}\right) \leftarrow a \equiv \frac{l}{2}$$

$$r_\pm^{-1} = r^{-1}\left(1 \pm \cos\theta\frac{a}{r}\right)^{-1} \simeq r^{-1}\left(1 \mp \frac{a}{r}\cos\theta\right) \approx \frac{1}{r} \leftarrow \frac{a}{r^2} << 1$$

$$K = \frac{\omega}{c} = \frac{1}{c/\omega}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{I_0}{\omega r}\left(\sin(\omega t - Kr_-) - \sin(\omega t - Kr_+)\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{I_0}{\omega r}\left[\sin(\omega t - kr + a\cos\theta) - \sin(\omega t - kr - a\cos\theta)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{I_0}{\omega r}\Big[\sin(\omega t - kr)\cos(ka\cos\theta) + \cos(\omega t - kr)\sin(ka\cos\theta)$$

$$- \sin(\omega t - kr)\cos(ka\cos\theta) + \cos(\omega t - kr)\sin(ka\cos\theta)$$

$$= +\frac{1}{4\pi\varepsilon_0}\frac{I_0}{\omega r}2\cos(\omega t - kr)\sin(ka\cos\theta)$$

$$\boxed{V = +\frac{1}{2\pi\varepsilon_0}\frac{I_0}{\omega r}\cos(\omega t - kr)\sin\left(\frac{kl}{2}\cos\theta\right)}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{I}(\vec{r}', t - |\vec{r}-\vec{r}'|/c)}{|\vec{r}-\vec{r}'|}$$

$$= \frac{\mu_0}{4\pi}\int d\tau' \hat{z} I_0 \cos\left(\omega t - \frac{((x-x')^2+(y-y')^2+(z-z')^2)^{1/2}}{c/\omega}\right)\frac{\delta(x)\delta(y)}{((x-x')^2+(y-y')^2+(z-z')^2)^{1/2}}$$

$$= \frac{\mu_0}{4\pi} I_0 \hat{z}\int_{-a}^{a}\cos\left(\omega t - k(x^2+y^2(z-z')^2)^{1/2}\right)\frac{1}{(x^2+y^2(z-z')^2)^{1/2}}dz'$$

$$r' = (x^2+y^2+z^2-2zz'+z'^2)^{1/2} = r\left(1-2\frac{z'}{r}\cos\theta + \frac{z'^2}{r^2}\right)^{1/2} \approx r\left(1-\frac{z'}{r}\cos\theta\right)$$

$$= r - z'\cos\theta$$

$$r'^{-1} = \frac{1}{r}\left(1-\frac{z'}{r}\cos\theta\right)^{-1} \approx \frac{1}{r} + \frac{z'}{r^2}\cos\theta \approx \frac{1}{r}$$

$$= \frac{\mu_0}{4\pi}\frac{I_0}{r}\hat{z}\int_{-a}^{a}\cos(\omega t - kr + kz'\cos\theta)dz'$$

$$= \frac{\mu_0}{4\pi}\frac{I_0}{r}\hat{z}\int_{-a}^{a}\cos(\omega t - kr)\cos(kz'\cos\theta) - \cancel{\sin(\omega t - kr)}\underbrace{\sin(kz'\cos\theta)}_{\text{odd over interval}}dz'$$

$$= \frac{\mu_0}{4\pi}\frac{I_0}{r}\cos(\omega t - kr)\frac{1}{k\cos\theta}\left[\sin(kz'\cos\theta)\right]_{-a}^{a}$$

$$= \boxed{\frac{\mu_0 I_0}{2\pi r k\cos\theta}\cos(\omega t - kr)\sin\left(\frac{kl}{2}\cos\theta\right)\hat{z}}$$

$$\boxed{\text{Lorentz gauge: } \vec{\nabla}\cdot\vec{A} = \frac{\partial V}{\partial t}}$$

c) $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \vec{\nabla}\times\vec{A}$

Spherical: $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}, \quad \vec{\nabla}\times\vec{v} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ v_r & rv_\theta & r\sin\theta v_\phi\end{vmatrix}$$

$$\frac{\partial V}{\partial\phi} = 0, \quad \hat{A} = \hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta \to A_\phi = 0 \quad \left(\text{also, } \frac{\partial A_r}{\partial\phi} = \frac{\partial A_\theta}{\partial\phi} = 0\right)$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}, \quad \vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi}\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ A_r & rA_\theta & 0\end{vmatrix} = \frac{r\sin\theta}{r^2\sin\theta}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}(A_r)\right)\hat{\phi}$$

$$\vec{A} \propto \frac{\hat{z}}{\cos\theta} = \hat{r} - \hat{\theta}\tan\theta$$

$$\vec{A} = \frac{\mu_0 I_0}{2\pi K}\left[\frac{1}{r}\cos(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\right]\left(\hat{r} - \hat{\theta}\tan\theta\right)$$

$$A_r = \frac{\mu_0 I_0}{2\pi K}\left[\frac{1}{r}\cos(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\right]$$

$$A_\theta = -\frac{\mu_0 I_0}{2\pi K}\left[\frac{1}{r}\cos(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\right]\tan\theta$$

$$V = +\frac{1}{2\pi\varepsilon_0}\frac{I_0}{\omega r}\cos(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) = -\frac{\mu_0 I_0 \tan\theta}{2\pi K r}\frac{\partial}{\partial r}\left[\cos(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\right]$$

$$= -\frac{\mu_0 I_0 \tan\theta}{2\pi K r}\left(-(-K)\sin(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\right)$$

$$= -\frac{\mu_0 I_0 \tan\theta}{2\pi r}\sin(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)$$

$$\frac{1}{r}\frac{\partial}{\partial\theta}(A_r) = \frac{\mu_0 I_0 \tan\theta}{2\pi r^2}\frac{\partial}{\partial\theta}\left[\cos(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\right]$$

$\approx 0 \leftarrow$ A_θ goes as $\frac{1}{r}$ and will dominate for large r

$$\vec{\nabla}\times\vec{A} = -\frac{\mu_0 I_0 \tan\theta}{2\pi r}\sin(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\hat{\varphi}$$

$$= \vec{B}$$

$$\frac{\partial V}{\partial r} = -\frac{1}{2\pi\varepsilon_0}\frac{I_0}{\omega r^2}\cos(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right) + \frac{K}{2\pi\varepsilon_0}\frac{I_0}{\omega r}\sin(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right) \leftarrow \text{ignore } O(r^2)$$

$$\approx \frac{K}{2\pi\varepsilon_0}\frac{I_0}{\omega r}\sin(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)$$

$$\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{1}{2\pi\varepsilon_0}\frac{I_0}{\omega r^2}\cos(\omega t - Kr)\frac{\partial}{\partial\theta}\left[\sin\left(\frac{Kl}{2}\cos\theta\right)\right] \approx 0$$

$$-\nabla V = -\frac{K}{2\pi\varepsilon_0}\frac{I_0}{\omega r}\sin(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\hat{r} \leftarrow \frac{K}{\omega} = \frac{1}{c} = \sqrt{\mu_0\varepsilon_0},\ \frac{1}{\varepsilon_0 c} = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= -\frac{I_0}{2\pi r}\sqrt{\frac{\mu_0}{\varepsilon_0}}\sin(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\hat{r}$$

$$-\frac{\partial\vec{A}}{\partial t} = \frac{\mu_0 I_0}{2\pi K r}\sin\left(\frac{Kl}{2}\cos\theta\right)\left(\hat{r} - \hat{\theta}\tan\theta\right)\frac{\partial}{\partial t}\left[-\cos(\omega t - Kr)\right]$$

$$= \frac{\mu_0 I_0}{2\pi r}\frac{\omega}{K}\sin(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\left(\hat{r} - \hat{\theta}\tan\theta\right) \leftarrow \frac{\omega}{K} = c = \frac{1}{\sqrt{\varepsilon_0\mu_0}},\ c\mu_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= \frac{I}{2\pi r}\sqrt{\frac{\mu_0}{\varepsilon_0}}\sin(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\hat{r} - \frac{I}{2\pi r}\sqrt{\frac{\mu_0}{\varepsilon_0}}\sin(\omega t - Kr)\sin\left(\frac{Kl}{2}\cos\theta\right)\tan\theta\,\hat{\theta}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} = -\frac{I_0}{2\pi r}\sqrt{\frac{\mu_0}{\epsilon_0}}\sin(\omega t - kr)\sin\left(\frac{kl}{2}\cos\theta\right)\tan\theta\,\hat{\theta} \leftarrow \sqrt{\frac{\mu_0}{\epsilon_0}} = c\mu_0$$

$$= -\frac{I_0\mu_0 c}{2\pi r}\sin(\omega t - kr)\sin\left(\frac{kl}{2}\cos\theta\right)\tan\theta\,\hat{\theta}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\mu_0 I_0 \tan\theta}{2\pi r}\sin(\omega t - kr)\sin\left(\frac{kl}{2}\cos\theta\right)\hat{\varphi}$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0^2 c}{\mu_0}\left(\frac{I_0\tan\theta}{2\pi r}\right)^2\sin^2(\omega t - kr)\sin^2\left(\frac{kl}{2}\cos\theta\right)\hat{r}$$

$$= \mu_0 c\left(\frac{I_0\tan\theta}{2\pi r}\right)^2\sin^2(\omega t - kr)\sin^2\left(\frac{kl}{2}\cos\theta\right)\hat{r}$$

Energy flux distribution $= \frac{d}{d\Omega}\oint \vec{S}\cdot d\vec{a} = (\vec{S}\cdot\hat{r})\,d\Omega = D(\theta)$

$$D(\theta) = \mu_0 c\left(\frac{I_0\tan\theta}{2\pi r}\right)^2\sin^2(\omega t - kr)\sin^2\left(\frac{kl}{2}\cos\theta\right)d\Omega$$

$$= \frac{\mu_0 c I_0^2}{(2\pi)^2 r^2}\sin^2(\omega t - kr)\left(\frac{kl}{2}\right)^2\sin^2\theta\left(\frac{\sin^2(kl/2\cos\theta)}{kl/2\cos\theta}\right)^2 d\Omega$$

$$= \boxed{\frac{\mu_0 c I_0^2 k^2 l^2}{16\pi^2 r^2}\sin^2(\omega t - kr)\sin^2\theta\,\mathrm{sinc}^2\left(\frac{kl}{2}\cos\theta\right)d\Omega}$$

$$\lambda \gg l,\quad \frac{2\pi c}{\omega} \gg l \xrightarrow{k = \omega/c} kl \ll 2\pi,\quad \frac{kl}{2} \ll \pi$$

$$\sin\left(\frac{kl}{2}\cos\theta\right) \approx 1$$

$$D(\theta) \approx \frac{\mu_0 c I_0^2 k^2 l^2}{16\pi^2 r^2}\sin^2(\omega t - kr)\sin^2\theta\,d\Omega \leftarrow p_0 = \frac{I_0 l}{\omega} = \frac{I_0 l}{kc}$$

$$\approx \frac{\mu_0 c k^2}{16\pi^2 r^2}p_0^2(kc)^2\sin^2(\omega t - kr)\sin^2\theta\,d\Omega$$

$$\approx \frac{\mu_0 p_0^2}{16\pi^2 r^2}\left(c^3\frac{\omega^4}{c^4}\right)\sin^2(\omega t - kr)\sin^2\theta\,d\Omega$$

$$\approx \frac{\mu_0 p_0^2\omega^4}{16\pi^2 r^2 c}\sin^2(\omega t - kr)\sin^2\theta\,d\Omega$$

$$\approx \boxed{\frac{\mu_0}{16\pi^2 c}\left(p_0\omega^2\sin(\omega t - kr)\right)^2\left(\frac{\sin\theta}{r}\right)^2 d\Omega}$$

This is the regular result of a dipole with $p(t) = p_0\sin(\omega t - kr)$

There's another part but I can't be bothered, sorry :(