Section B. Electricity and Magnetism

1. Conducting plane with bulge

(a) A spherical conductor of radius \(a\) is at potential \(V = 0\) with respect to infinity. A charge \(Q = q\) is brought to a distance \(p > a\) from the center of the sphere and you are asked to find the force on the charge. Show that this can be determined with the help of a notional image charge \(Q' = -\frac{a}{p}q\) located a distance \(\frac{a^2}{p}\) from the center of the sphere.

(b) Use what you have learned in a) about image charges in a sphere to analyze the following more complicated situation: A conductor at potential \(V = 0\) has the shape of an infinite plane except for a hemispherical bulge of radius \(a\). A charge \(q\) is placed above the center of the bulge, a distance \(p\) from the place (distance \(p - a\) from the top of the bulge). What is the force on the charge?
a) Image charge in sphere

\[ \vec{r} = \vec{x} - \rho \vec{d} = (a \sin \theta \cos \phi - \rho) \hat{x} + (a \sin \theta \sin \phi) \hat{y} + (\rho \cos \phi) \hat{z} \]

\[ \rho^2 = \rho^2 \sin^2 \theta + \rho^2 - 2 \rho \rho \sin \theta \cos \phi + a^2 \sin^2 \theta + a^2 \cos^2 \theta \]

\[ \overrightarrow{C} = \rho \overrightarrow{d} \]

\[ C^2 = a^2 + d^2 - 2a \rho \sin \theta \cos \phi \]

\[ V(\text{surface}) = \frac{1}{4\pi \epsilon_0} \left( \frac{q}{r} + \frac{q'}{r'} \right) = 0 \rightarrow q' r = C q \rightarrow q'^2 r^2 = q^2 C^2 \]

\[ q'^2 \left( a^2 + \rho^2 - 2 \rho \rho \sin \theta \cos \phi \right) = q^2 \left( a^2 + d^2 - 2\rho \rho \sin \theta \cos \phi \right) \rightarrow \text{must be valid for all } \theta \text{ and } \phi \]

\[ \begin{cases} q'^2 (a^2 + \rho^2) = q^2 \left( a^2 + d^2 \right) \\ q'^2 (2 \rho \rho \sin \theta \cos \phi) = q^2 (2 \rho \rho \sin \theta \cos \phi) \end{cases} \Rightarrow \frac{q'}{q} = \frac{\rho}{\rho} \]

\[ q^2 \left( a^2 + d^2 \right) = q'^2 \left( a^2 + \rho^2 \right) = q^2 \left( a^2 + \rho^2 \right) \]

\[ d^2 + a^2 = \left( \frac{a^2}{r} + \rho \right) d \rightarrow d^2 - \left( \frac{a^2}{r} + \rho \right) d + a^2 = 0 \]

\[ d = \frac{\left( \frac{a^2}{r} + \rho \right) \pm \sqrt{\left( \frac{a^2}{r} + \rho \right)^2 - 4 \frac{a^2}{r}}}{2} = \frac{1}{2} \left[ \left( \frac{a^2}{r} + \rho \right) \pm \sqrt{\frac{a^2}{r} + \rho} \right] = \frac{a^2}{r} \text{ or } \rho \]

\[ q'^2 = q^2 \frac{d}{\rho} \rightarrow q' = \pm q \frac{\rho}{\rho} = \Theta \frac{q^2}{\rho} \]

\[ d = \frac{a^2}{r} \]

\[ q' = \frac{-a}{r} q \]
6) Gravitational charge continuity

\[ \vec{q} \times \vec{p} \rightarrow 0 \quad \text{at} \quad \theta = \theta' \quad \rightarrow \quad 0 \quad \text{at} \quad \theta = \theta' \]

\[ \sigma \rightarrow \frac{1}{r^2} \]

\[ \vec{E}(\text{at } \theta) = \frac{1}{4\pi\varepsilon_0} \left[ \frac{-q \hat{r}}{(r^2 - a^2)^2} \left( \frac{r^2 + a^2}{2a^2} \right) \right] \]

\[ = \frac{1}{4\pi\varepsilon_0} \left[ \frac{-q r^2}{(r^2 + a^2)^2} \right] \]

\[ = \frac{1}{4\pi\varepsilon_0} \left[ \frac{-q a^2}{(r^2 + a^2)^2} \right] 

\[ = \frac{9}{16\pi\varepsilon_0} \left[ \frac{16a^2 p^2 + a^8 - 2a^4 p^2}{p^2 + a^8 - 2a^4 p^2} \right] \]

\[ \vec{F} = q \vec{E}(\text{at } \theta) = \frac{9}{16\pi\varepsilon_0} \left[ \frac{16a^3 p^5 + a^9 - 2a^5 p^5}{p^2 + a^8 - 2a^4 p^2} \right] \]

\[ \int = \frac{9}{16\pi\varepsilon_0} \left[ \frac{16a^3 p^5 + a^9 - 2a^5 p^5}{p^2 + a^8 - 2a^4 p^2} \right] \]