Prelims Solutions

Problem J15T3

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1

In the high temperature limit the BE distribution is closely approximated by $f(\epsilon)_{BE} = \frac{1}{\lambda^{-1}e^{\beta\epsilon}-1} \approx \lambda e^{-\beta\epsilon}$. For $3D$ spin 0 particles, our density of states is $D(\epsilon) = 1/8 * 4\pi * 1 * (\frac{L}{c\hbar\pi})^3\epsilon^2 = A\epsilon^2$ (they set $\hbar = 1$). Now write out N and U:

$$N = \int_0^\infty D(\epsilon)f(\epsilon)_{BE}d\epsilon = A\lambda\int_0^\infty \epsilon^2 e^{-\beta\epsilon}d\epsilon = A\lambda(k_BT)^3\int_0^\infty x^2e^{-x}dx$$

$$U = \int_0^\infty D(\epsilon)\epsilon f(\epsilon)_{BE}d\epsilon = A\lambda\int_0^\infty \epsilon^3 e^{-\beta\epsilon}d\epsilon = A\lambda(k_BT)^4\int_0^\infty x^3e^{-x}dx = 3A\lambda(k_BT)^4\int_0^\infty x^2e^{-x}dx$$

using integration by parts. Hence, $U/N = 3k_BT$.

2

Below the critical temperature the chemical potential is 0 so $\mu = \frac{\partial F}{\partial N} = 0$ means F is independent of N. So the pressure $P = -\frac{\partial F}{\partial V}$ is also independent of N below the critical temperature.

Below the critical temperature the gas forms a condensate with a macroscopic number of particles that are forced into the ground state because all the excited states are thermally saturated. At the critical temperature all the excited state are just filled. This occurs at the critical density that forces $\lambda = 1$ (and λ stays equal to 1 below the critical temperature).

$$N = \int_0^\infty D(\epsilon)f(\epsilon, \lambda = 1)_{BE}d\epsilon = A(k_bT)^3\int_0^\infty \frac{x^2}{e^x - 1}dx = A(k_bT)^3I_1$$

where I_1 is some numerical value O(1).
So $T_c = (\frac{N}{AI_1})^{\frac{1}{3}}/k_b = \frac{c\hbar\pi}{k_b}(\frac{2N}{\pi I_1 V})^{\frac{1}{3}}$ using $L^3 = V$.

3

We need to find U below T_c.

$$U = \int_0^\infty D(\epsilon)\epsilon f(\epsilon, \lambda = 1)_{BE}d\epsilon = A(k_bT)^4\int_0^\infty \frac{x^3}{e^x - 1}dx = A(k_bT)^4I_2$$

Hence,

$$C_v = \frac{dU}{dT}|_V = 4A(k_bT)^3I_2 = 2\pi I_2\frac{V}{(c\hbar\pi)^3}(k_bT)^3$$

4

In 2D the density of states is linear with n and thus ϵ. The integral for N at $\lambda = 1$ becomes $\propto \int_0^\infty \frac{x}{e^x-1}dx$ which converges, so a BEC will need to form to accommodate particles at higher N or lower T.