Prelims Solutions

Problem J15T3

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1

In the linear approximation the thermal current $J = -k\frac{dT}{dz}$. Conservation of heat/energy requires something like $\frac{\partial Q}{\partial t} = -\frac{\partial J}{\partial z}$ where $dQ = dU = C_V T$. Combining these gives $\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial z^2}$. Try a solution of the form $T(z,t) \propto e^{ikz+i\omega t}$ and hope for the best.
This gives $i\omega = -Dk^2$, forcing $k = \pm(-1+i)\sqrt{\frac{\omega}{2D}}$ for a given ω. We take ω to be real since the boundary conditions are sinusoidal with time and $k = -(-1+i)\sqrt{\frac{\omega}{2D}}$ since the temperature goes to T_o as $z \to -\infty$. So we see that the spatial part of $T(z,t)$ will have a cosine and exponential dependence, cosine because we have nonzero T at $z = 0$. Hence,

$$T(z,t) = T_0 + T_a cos(\omega_a t) cos(\sqrt{\frac{\omega_a}{2D}} z) e^{\sqrt{\frac{\omega_a}{2D}} z} + T_d cos(\omega_d t) cos(\sqrt{\frac{\omega_d}{2D}} z) e^{\sqrt{\frac{\omega_d}{2D}} z}$$

which satisfies the boundary conditions and the diffusion equation.

2

This occurs when the annual part of the temperature $T_a cos(\omega_a t) cos(\sqrt{\frac{\omega_a}{2D}} z) e^{\sqrt{\frac{\omega_a}{2D}} z}$ picks up a negative sign relative to the surface $T_a cos(\omega_a t)$:

$$cos(\sqrt{\frac{\omega_a}{2D}} z) = -1 \to z_{opposite} = \pi\sqrt{\frac{2D}{\omega_a}}$$

3

$$e^{-\sqrt{\frac{\omega_a}{2D}} z_{opposite}} = e^{-\pi}$$

which almost equals -1.