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Consider a toy model of the Helium atom where the Coulombic interaction potential is replaced with a Hooke Law's potential. If the nucleus of the atom is located at $\vec{\mathbf{r}} = 0$ and the electrons of mass m have position vectors $\vec{\mathbf{r_1}}$ and $\vec{\mathbf{r_2}}$ the interaction potential is

$$V(\vec{\mathbf{r_1}}, \vec{\mathbf{r_1}}) = \frac{1}{2}m\omega^2(\vec{\mathbf{r_1}}^2 + \vec{\mathbf{r_2}}^2) - \frac{\lambda}{4}m\omega^2(\vec{\mathbf{r_1}} - \vec{\mathbf{r_2}})^2$$

This model is exactly solvable. Assume $\lambda > 0$

1 Part A

What constraint must be imposed on λ for the system to be well-behaved? *[Hint: It may be useful to consider the center of mass and relative position vectors of the two electrons* $\vec{\mathbf{u}} = \frac{\vec{\mathbf{r_1}}+\vec{\mathbf{r_2}}}{2}$ *and* $\vec{\mathbf{v}} = \vec{\mathbf{r_1}} - \vec{\mathbf{r_2}}$*.]*

As is suggested we rewrite the potential in terms of the center of mass and relative position vectors

$$V(\vec{\mathbf{u}}, \vec{\mathbf{v}}) = m\omega^2\vec{\mathbf{u}}^2 + \frac{1}{4}m\omega^2(1-\lambda)\vec{\mathbf{v}}^2$$

We note that for certain values of λ that the Hamiltonian will look like two 3D harmonic oscillators when both terms have quadratic potential terms have positive coefficients. The condition on λ for this to be true is that

$$0 < \lambda < 1$$

2 Part B

What are the energy levels of the system when $\lambda = \frac{1}{2}$?

We can write the total Hamiltonian in center of mass and relative position coordinate vectors and the corresponding momentum vectors:

$$H = \frac{\vec{\mathbf{P_u}}^2}{4m} + \frac{\vec{\mathbf{P_v}}^2}{m} + m\omega^2\vec{\mathbf{u}}^2 + \frac{1}{8}m\omega^2\vec{\mathbf{v}}^2$$

The denominators of the kinetic parts of the Hamiltonian come from the fact that the total mass is $2m$ and the reduced mass is $\frac{m}{2}$. Now we note that the general form of the Hamiltonian of a harmonic oscillator is:

$$H = \frac{\vec{\mathbf{P}}^2}{2m} + \frac{1}{2}m\omega^2\vec{\mathbf{r}}^2$$

Thus we can rewrite our Hamiltonian in terms of frequencies ω_u and ω_v which can be determined as follows:

$$\frac{1}{2}2m\omega_u^2 = m\omega^2, \omega_u = \omega$$
$$\frac{1}{2}\frac{m}{2}\omega_v^2 = \frac{1}{8}m\omega^2, \omega_v = \frac{1}{\sqrt{2}}\omega$$

Our energy levels are now given by the sum of two 3D harmonic oscillators with frequencies ω_u and ω_v

$$E_{n_1,n_2,n_3,m_1,m_2,m_3} = \hbar\omega_u(n_1 + n_2 + n_3 + \frac{3}{2}) + \hbar\omega_v(m_1 + m_2 + m_3 + \frac{3}{2})$$

3 Part C

Taking into account the spin of the electrons, what are the degeneracies of the lowest four energy levels when $\lambda = \frac{1}{2}$?

The lowest four energy levels will be:

$$E_0 = \frac{3}{2}\hbar\omega + \frac{3}{2}\hbar\frac{\omega}{\sqrt{2}}$$
$$E_1 = \frac{3}{2}\hbar\omega + \frac{5}{2}\hbar\frac{\omega}{\sqrt{2}}$$
$$E_2 = \frac{5}{2}\hbar\omega + \frac{3}{2}\hbar\frac{\omega}{\sqrt{2}}$$
$$E_3 = \frac{3}{2}\hbar\omega + \frac{7}{2}\hbar\frac{\omega}{\sqrt{2}}$$

The degeneracy of each of the energy levels are 1, 3, 3, and 6 respectively.

4 Part D

Suppose the Helium atom is initially in it's third excited state. It then undergoes a decay through an electric dipole transition to a lower-energy state. What are the possible energies of the emitted photons?

The possible energies will be:

$$E_3 - E_0 = \sqrt{2}\hbar\omega$$
$$E_3 - E_1 = \frac{1}{\sqrt{2}}\hbar\omega$$
$$E_3 - E_2 = (\sqrt{2} - 1)\hbar\omega$$