PROBLEM J15Q.2

(a) The leading-order coupling between the $2s$ and $2p$ states comes from the electric dipole term at $\mathbf{r} = 0$. The higher-order terms (magnetic dipole, electric quadrupole, etc.) are suppressed by a factor of $(a_B/b)^2$.

Thus we need only compute

$$\mathbf{E}|_{\mathbf{r}=0} = -\frac{Qe}{R^3}\mathbf{R},$$

where $\mathbf{R} := \mathbf{b} + \mathbf{v}t$, and make the linear (dipole) approximation

$$V_{\text{dip}}(\mathbf{r}, t) \sim \frac{Qe}{R^3}\mathbf{R} \cdot \mathbf{r} = \frac{Qe}{(b^2 + v^2t^2)^{3/2}}(bz + vty).$$

(b) Since the $2s$ and $2p$ states are assumed to be degenerate, the transition probability in first-order perturbation theory is

$$P_{i\to f} = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} \langle f|V_{\text{dip}}|i\rangle \, \mathrm{d}t\right|^2,$$

where the probability should be summed over all final states.

Note that V_{dip} is a linear combination of the y and z operators. By symmetry, we have

$$\langle \phi_{2p,\pm 1}|z|\phi_{2s}\rangle = 0 \quad \text{and} \quad \langle \phi_{2p,0}|y|\phi_{2s}\rangle = 0.$$

In particular, it follows that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \langle \phi_{2p,\pm 1}|V_{\text{dip}}|\phi_{2s}\rangle \, \mathrm{d}t \\
&= Qe\,\langle \phi_{2p,\pm 1}|y|\phi_{2s}\rangle \int_{-\infty}^{\infty} \frac{vt}{(b^2 + v^2t^2)^{3/2}}\,\mathrm{d}t \\
&= 0,
\end{aligned}$$

and so $P_{2s\to 2p,\pm 1} = 0$.

Thus the only relevant nonzero matrix element is

$$\langle \phi_{2p,0}|z|\phi_{2s}\rangle = \frac{1}{16\pi a_B^4}\int z^2 e^{-r/a_B}\left(1 - \frac{r}{2a_B}\right)\mathrm{d}V\,.$$

By spherical symmetry, the integral is equal to

$$\begin{aligned}
\frac{1}{3}\int r^2 e^{-r/a_B}\left(1 - \frac{r}{2a_B}\right)\mathrm{d}V &= \frac{4\pi}{3}\int_0^{\infty} r^4 e^{-r/a_B}\left(1 - \frac{r}{2a_B}\right)\mathrm{d}r \\
&= \frac{4\pi}{3}a_B^5\int_0^{\infty} u^4 e^{-u}\left(1 - \frac{u}{2}\right)\mathrm{d}u \\
&= \frac{4\pi}{3}a_B^5(4! - 5!/2) \\
&= -48\pi a_B^5,
\end{aligned}$$

where $u := r/a_B$. We conclude that

$$\langle \phi_{2p,0}|z|\phi_{2s}\rangle = -3a_B,$$

and so

$$\begin{aligned}P_{2s\to 2p} = P_{2s\to 2p,\pm 0} &= \frac{(3a_B Qeb)^2}{\hbar^2}\left|\int_{-\infty}^{\infty} \frac{1}{(b^2+v^2t^2)^{3/2}}\,\mathrm{d}t\right|^2 \\ &= \frac{9a_B^2 Q^2 e^2 b^2}{\hbar^2}\frac{1}{b^4 v^2}\left|\int_{-\infty}^{\infty} \frac{1}{(1+u^2)^{3/2}}\,\mathrm{d}u\right|^2 \\ &= \boxed{\frac{36 a_B^2 Q^2 e^2}{\hbar^2 b^2 v^2},}\end{aligned}$$

where $u := vt/b$.