J15M.3 (Solution by Jim Wu)
A uniform cylinder of mass m and radius b rolls off a fixed cylindrical surface of radius R under the influence of gravity. The axes of both cylinders are horizontal. The rolling cylinder starts from the top of the fixed cylinder with negligibly small velocity.

(a) If we assume the cylinder rolls without slipping, find the angle θ from the vertical when it loses contact with the fixed cylinder.

(b) In practice for a finite value of μ the cylinder will start to slip before it loses contact. Find the angle when it starts to slip for $\mu = 1$.

Solution:

(a) From Newton's second law,

$$N - mg\cos\theta = -\frac{mv^2}{R+b}$$
$$mg\sin\theta - F = ma_\theta$$

where F is the force of friction. From energy conservation,

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mg(R+b)\cos\theta = mg(R+b)$$
$$\frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{b}\right)^2 = mg(R+b)(1-\cos\theta)$$
$$v^2 = \frac{2g(R+b)(1-\cos\theta)}{1+\beta}$$

where $I = \beta mb^2$. At the point where the rolling cylinder loses contact with the fixed cylinder, $N \to 0$. The angle at which it happens is

$$mg\cos\theta = \frac{m}{R+b}\left(\frac{2g(R+b)(1-\cos\theta)}{1+\beta}\right)$$
$$\cos\theta = \frac{2(1-\cos\theta)}{1+\beta}$$
$$\cos\theta = \frac{2}{3+\beta}$$

For a cylinder, $\beta = \frac{1}{2}$, so the angle is

$$\theta = \cos^{-1}\left(\frac{4}{7}\right) \approx 55.15^\circ$$

(b) The condition for slipping is when the forward center of mass acceleration is less than the acceleration at the point of contact due to angular acceleration, or

$$a_\theta < b\alpha$$

This happens when the static friction becomes maximal, $F = \mu N$! From torque,

$$I\alpha = Fb$$

and substituting the torque equation and Newton's second law into the inequality gives

$$\begin{aligned} mg\sin\theta - F &< \frac{Fmb^2}{I} \\ mg\sin\theta &< \left(\frac{1+\beta}{\beta}\right)F \end{aligned}$$

The frictional force is given by

$$\begin{aligned} F = \mu N &= \mu\left(mg\cos\theta - \frac{mv^2}{R+b}\right) \\ &= \mu m\left(g\cos\theta - \frac{2g(1-\cos\theta)}{1+\beta}\right) \\ &= \mu mg\left(\frac{3+\beta}{1+\beta}\cos\theta - \frac{2}{1+\beta}\right) \end{aligned}$$

Substituting this back into the inequality, we have

$$\begin{aligned} \mu\left(\frac{1+\beta}{\beta}\right)\left(\frac{3+\beta}{1+\beta}\cos\theta - \frac{2}{1+\beta}\right) &> \sin\theta \\ \mu[(3+\beta)\cos\theta - 2] &> \beta\sin\theta \\ \mu(3+\beta)\cos\theta - \beta\sin\theta &> 2 \end{aligned}$$

To solve this equation, let us simplify the left hand side by using Euler's formula for the trigonometric functions:

$$\begin{aligned} \mu(3+\beta)\cos\theta - \beta\sin\theta &= \frac{1}{2}\left[\mu(3+\beta)(e^{i\theta}+e^{-i\theta}) + i\beta(e^{i\theta}-e^{-i\theta})\right] \\ &= \frac{1}{2}\left[(\mu(3+\beta)+i\beta)e^{i\theta} + (\mu(3+\beta)-i\beta)e^{-i\theta}\right] \\ &= \frac{\sqrt{\mu^2(3+\beta)^2+\beta^2}}{2}\left[e^{i\left(\theta+\tan^{-1}\frac{\beta}{\mu(3+\beta)}\right)} + e^{-i\left(\theta+\tan^{-1}\frac{\beta}{\mu(3+\beta)}\right)}\right] \\ &= \sqrt{\mu^2(3+\beta)^2+\beta^2}\cos\left(\theta+\tan^{-1}\left(\frac{\beta}{\mu(3+\beta)}\right)\right) \end{aligned}$$

Therefore, the angle at which it starts to slip is

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{\mu^2(3+\beta)^2+\beta^2}}\right) - \tan^{-1}\left(\frac{\beta}{\mu(3+\beta)}\right)$$

For $\mu = 1$ and $\beta = \frac{1}{2}$, this angle is

$$\theta = \cos^{-1}\left(\frac{4}{\sqrt{50}}\right) - \tan^{-1}\left(\frac{1}{7}\right) \approx 47.42^\circ$$

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