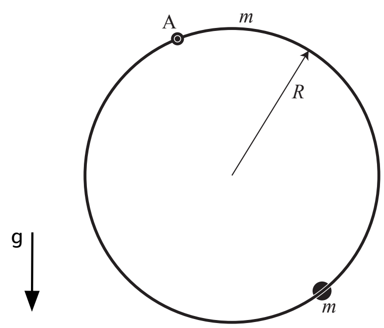
4 Dec 2022

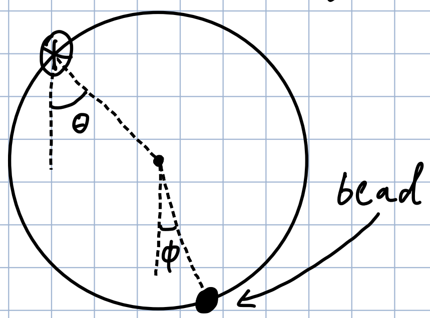
2. A thin hoop of mass m and radius R is suspended from its rim (point A) and is free to rotate around point A in the plane of the hoop. A small bead of equal mass m can slide without friction on the hoop.



Find the frequencies of the normal modes for this system.

J15M.2

Subscripts: r = ring, b = bead



$$x_b = R(\sin\theta + \sin\phi)$$
$$y_b = -R(\cos\theta + \cos\phi)$$
$$\dot{x}_b = R(\dot\theta\cos\theta + \dot\phi\cos\phi)$$
$$\dot{y}_b = R(\dot\theta\sin\theta + \dot\phi\sin\phi)$$

$$\dot{x}_b^2 = R^2(\dot\theta^2\cos^2\theta + \dot\phi^2\cos^2\phi + 2\dot\phi\dot\theta\cos\theta\cos\phi)$$
$$\dot{y}_b^2 = R^2(\dot\theta^2\sin^2\theta + \dot\phi^2\sin^2\phi + 2\dot\phi\dot\theta\sin\theta\sin\phi)$$

Plan: Find EoMs, assume small angle, and find the eigenmodes

$$L = T - V, \quad T = T_r + T_b, \quad V = V_r + V_b$$

$$T_r = \frac{1}{2} I_r \dot\theta^2 \leftarrow I_r = \underbrace{MR^2}_{\text{intrinsic}} + \underbrace{MR^2}_{\text{parallel axis theorem}} = 2MR^2$$
$$= mR^2\dot\theta^2$$

$$T_b = \frac{1}{2} m(\dot{x}_b^2 + \dot{y}_b^2) = \frac{1}{2} mR^2(\dot\theta^2\cos^2\theta + \dot\phi^2\cos^2\phi + 2\dot\phi\dot\theta\cos\theta\cos\phi + \dot\theta^2\sin^2\theta + \dot\phi^2\sin^2\phi + 2\dot\phi\dot\theta\sin\theta\sin\phi)$$
$$= \frac{1}{2} mR^2(\dot\theta^2 + \dot\phi^2 + 2\dot\phi\dot\theta(\cos\theta\cos\phi + \sin\theta\sin\phi)) \leftarrow \cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta - \phi)$$
$$= \frac{1}{2} MR^2(\dot\theta^2 + \dot\phi^2 + 2\phi\dot\theta\cos(\theta - \phi))$$

$$T = \frac{1}{2} mR^2(3\dot\theta^2 + \dot\phi^2 + 2\phi\dot\theta\cos(\theta - \phi))$$

$$V_r = -mgR\cos\theta$$

$$V_b = mgz = -mgR(\cos\theta + \cos\phi)$$

$$V = -mgR(2\cos\theta + \cos\phi)$$

$$L = \frac{1}{2} mR^2(3\dot\theta^2 + \dot\phi^2 + 2\phi\dot\theta\cos(\theta - \phi)) + mgR(2\cos\theta + \cos\phi)$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2} mR^2\left(-2\dot\theta\dot\phi\sin(\theta-\phi)\right) - mgR(2\sin\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{d}{dt}\left(\frac{1}{2} mR^2\left(6\dot\theta + 2\dot\phi\cos(\theta-\phi)\right)\right) = mR^2\left(3\ddot\theta + \ddot\phi\cos(\theta-\phi) - \dot\phi\sin(\theta-\phi)(\dot\theta-\dot\phi)\right)$$

$$\frac{\partial L}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) \rightarrow$$

$$\frac{1}{2} mR^2\left(-2\dot\theta\dot\phi\sin(\theta-\phi)\right) - mgR(2\sin\theta) = mR^2\left(3\ddot\theta + \ddot\phi\cos(\theta-\phi) - \dot\phi\sin(\theta-\phi)(\dot\theta-\dot\phi)\right)$$

$\downarrow$ Small angles $\rightarrow$ $\phi, \theta \ll 1$, ignore $\mathcal{O}(\dot\theta^2, \dot\phi^2, \dot\phi\dot\theta)$ or higher

$$-2mgR\theta = mR^2(3\ddot\theta + \ddot\phi)$$

$$3\ddot\theta + \ddot\phi = -2\frac{g}{R}\theta \leftarrow \frac{g}{R} \equiv \omega_g^2$$

$$= -2\omega_g^2\theta$$

$$\frac{\partial L}{\partial \phi} = \frac{1}{2} mR^2\left(2\dot\theta\dot\phi(\theta-\phi)\right) - mgR(\sin\phi)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{d}{dt}\left(mR^2\left(\dot\phi + \dot\theta\cos(\theta-\phi)\right)\right) = mR^2\left(\ddot\phi + \ddot\theta\cos(\theta-\phi) - \dot\theta\sin(\theta-\phi)(\dot\theta-\dot\phi)\right)$$

$$\frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) \rightarrow$$

$$\frac{1}{2} mR^2\left(2\dot\theta\dot\phi(\theta-\phi)\right) - mgR(\sin\phi) = mR^2\left(\ddot\phi + \ddot\theta\cos(\theta-\phi) - \dot\theta\sin(\theta-\phi)(\dot\theta-\dot\phi)\right)$$

$\downarrow$ Small angles

$$-mgR\phi = mR^2(\ddot\phi + \ddot\theta)$$

$$\ddot\theta + \ddot\phi = -\omega_g^2\phi$$

$$\left.\begin{aligned} 3\ddot\theta + \ddot\phi &= -2\omega_g^2\theta \\ \ddot\theta + \ddot\phi &= -\omega_g^2\phi \end{aligned}\right\} \overbrace{\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}}^{\text{"M"}} \overbrace{\begin{pmatrix} \ddot\theta \\ \ddot\phi \end{pmatrix}}^{\ddot\eta} = -\omega_g^2 \overbrace{\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}}^{\text{"K"}} \overbrace{\begin{pmatrix} \theta \\ \phi \end{pmatrix}}^{\eta} \rightarrow M\ddot\eta = -K\eta$$

$|K - \omega^2 M| = 0 \leftarrow$ find eigenmodes

$$\begin{vmatrix} 2\omega_g^2 - 3\omega^2 & -\omega^2 \\ -\omega^2 & \omega_g^2 - \omega^2 \end{vmatrix} = 0 = (2\omega_g^2 - 3\omega^2)(\omega_g^2 - \omega^2) + (\omega^2)^2 = 2(\omega^2)^2 - 5\omega_g^2\omega^2 + 2(\omega_g^2)^2$$

$$\omega^2 = \omega_g^2 \frac{5 \pm \sqrt{25-16}}{2} = \omega_g^2 \frac{5 \pm 3}{2} = 2\omega_g^2, \tfrac{1}{2}\omega_g^2$$

$$\boxed{\omega = \sqrt{2}\,\omega_g \text{ and } \sqrt{\tfrac{1}{2}}\,\omega_g \,,\; \omega_g \equiv \sqrt{\tfrac{g}{R}}}$$