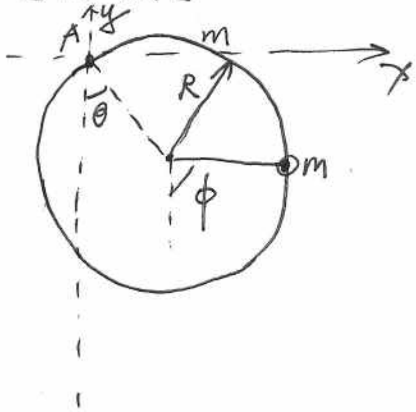
J15 M2



A is pivot.

Bead slides with no friction.

Find: Normal modes.

Solution:

let pivot be $(0, 0)$.

$$\begin{cases} x_{ring} = R\sin\theta; \\ y_{ring} = -R\cos\theta; \\ x_b = x_{ring} + R\sin\phi = R(\sin\theta + \sin\phi) \\ y_b = y_{ring} - R\cos\phi = -R(\cos\theta + \cos\phi). \end{cases}$$

Translational K.E:

$$T_t = \frac{1}{2}m(\dot{x}_{ring}^2 + \dot{y}_{ring}^2) + \frac{1}{2}m(\dot{x}_b^2 + \dot{y}_b^2)$$

$$= \frac{1}{2}m\left[R^2\dot{\theta}^2 + R^2(\cos^2\theta\,\dot{\theta}^2 + \sin^2\theta\,\dot{\theta}^2 + \cos^2\phi\,\dot{\phi}^2 + \sin^2\phi\,\dot{\phi}^2 + 2\cos\theta\cos\phi\,\dot{\theta}\dot{\phi} + 2\sin\theta\sin\phi\,\dot{\theta}\dot{\phi})\right]$$

$$= \frac{1}{2}mR^2\left[\dot{\theta}^2 + \dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right]$$

$$= \frac{1}{2}mR^2\left[2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\right]$$

Rotational K.E:

$$T_R = \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}(mR^2 + \cancel{mR^2})\dot{\theta}^2 = mR^2\dot{\theta}^2.$$

(parallel axis)

This term is redundant. The rotational energy should only contain the rotation around CM.
I'll leave it to the readers to fix the factors of 2…

Potential:

$$U = mg\,y_{ring} + mg\,y_b = -mgR(2\cos\theta + \cos\phi).$$

Therefore, Lagrangian is:

$$\boxed{\mathcal{L} = \frac{1}{2} m R^2 \left[4\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi\cos(\theta-\phi) \right] + mgR(2\cos\theta + \cos\phi)}.$$

Now find eq' of motion:

in θ:
$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \theta} = -2mgR\sin\theta - mR^2\dot\theta\dot\phi\sin(\theta-\phi) . \\ \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot\theta}\right) = \dfrac{1}{2} mR^2 \left[8\ddot\theta + 2\ddot\phi\cos(\theta-\phi) - 2\dot\phi\sin(\theta-\phi)\dot\theta + 2\dot\phi^2\sin(\theta-\phi) \right] . \end{cases}$$

$$\Rightarrow \underline{-2mgR\sin\theta = mR^2\left[4\ddot\theta + \ddot\phi\cos(\theta-\phi) + \dot\phi^2\sin(\theta-\phi) \right]} .$$

in ϕ:
$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \phi} = -mgR\sin\phi + mR^2\dot\theta\dot\phi\sin(\theta-\phi) \\ \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot\phi}\right) = mR^2\left[\ddot\phi + \ddot\theta\cos(\theta-\phi) - \dot\theta^2\sin(\theta-\phi) + \dot\theta\dot\phi\sin(\theta-\phi) \right] . \end{cases}$$

$$\Rightarrow \underline{-mgR\sin\phi = mR^2\left[\ddot\phi + \ddot\theta\cos(\theta-\phi) - \dot\theta^2\sin(\theta-\phi) \right]} .$$

To 1st order in θ & ϕ:

$$\begin{cases} -2g\theta = R\left[4\ddot\theta + \ddot\phi \right] & (1) \\ -g\phi = R\left[\ddot\phi + \ddot\theta \right] & (2) \end{cases}$$

$(1) - (2) \Rightarrow R3\ddot\theta = -2g\theta + g\phi \quad \Rightarrow \ddot\theta = \frac{g}{3R}(-2\theta + \phi)$.

$(1) - 4\times(2) \Rightarrow -3R\ddot\phi = -2g\theta + 4g\phi \quad \Rightarrow \ddot\phi = \frac{g}{3R}(2\theta - 4\phi)$.

$$\Rightarrow \begin{pmatrix} \ddot\theta \\ \ddot\phi \end{pmatrix} = \frac{g}{3R}\begin{pmatrix} -2 & 1 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} \theta \\ \phi \end{pmatrix} .$$

let $\begin{pmatrix} \theta(t) \\ \phi(t) \end{pmatrix} = \begin{pmatrix} A_\theta \\ A_\phi \end{pmatrix} e^{i\omega t}$ let $-\omega^2 = \frac{g}{3R}\lambda$. $\Rightarrow \lambda$ is eigenvalue of matrix.

$(-2-\lambda)(-4-\lambda) - 2 = 0$

$\lambda^2 + 6\lambda + 6 = 0$

$\lambda = -3 \pm \sqrt{3} \quad \Rightarrow \boxed{\begin{aligned} \omega_1 &= \sqrt{\tfrac{g}{3R}}\,(-3+\sqrt{3})^{1/2} \\ \omega_2 &= \sqrt{\tfrac{g}{3R}}\,(-3-\sqrt{3})^{1/2} \end{aligned}}$

when $\lambda = -3+\sqrt{3}$,

$$\begin{pmatrix} 1-\sqrt{3} & 1 \\ 2 & -1-\sqrt{3} \end{pmatrix}\begin{pmatrix} A_\theta \\ A_\phi \end{pmatrix} = 0 \quad \Rightarrow \begin{pmatrix} A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} 1 \\ 1-\sqrt{3} \end{pmatrix}.$$

when $\lambda = -3-\sqrt{3}$,

$$\begin{pmatrix} 1+\sqrt{3} & 1 \\ 2 & -1+\sqrt{3} \end{pmatrix}\begin{pmatrix} A_\theta \\ A_\phi \end{pmatrix} = 0 \quad \Rightarrow \begin{pmatrix} A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} 1 \\ 1+\sqrt{3} \end{pmatrix}.$$

Therefore, the two modes are:

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 1 \\ 1-\sqrt{3} \end{pmatrix} e^{i\left[\sqrt{\frac{g}{3R}}(-3+\sqrt{3})^{1/2} t - \delta_1\right]}, \text{ and}$$

$$\begin{pmatrix} \theta \\ \phi \end{pmatrix} = \begin{pmatrix} 1 \\ 1+\sqrt{3} \end{pmatrix} e^{i\left[\sqrt{\frac{g}{3R}}(-3-\sqrt{3})^{1/2} t - \delta_2\right]}.$$

where δ_1 & δ_2 are phases. (constants of integration).