Section A. Mechanics

1. (Precession due the Sun's oblateness) In addition to the celebrated relativistic effect, the precession of the perihelion of Mercury can be affected by the deviation in the Sun's mass distribution from spherical symmetry, caused by Sun's rotation around its axis. The distortion creates a small correction to the gravitational potential, which along the plane perpendicular to the axis of rotation is:

$$\delta V(r) = -\lambda \frac{mMG}{r^n}$$

where G is the gravitational constant, M and m are the masses of the Sun and Mercury respectively, λ is a small (dimensionfull) parameter, and the power is $n > 1$.

(a) Calculate the orbital angular velocity, ω_{orb}, for an orbit which is close to circular, at radius r, and lies within the plane perpendicular to the axis.

(b) Write down the effective potential for radial motion and calculate the frequency of small oscillations about its minimum, ω_{osc}, in terms r and the variables defined above.

(c) From (a) and (b) find the approximate rate of precession of the perihelion (the point of closest approach) for a slightly elliptical orbit with an average radius r.

(d) Depending on the sign of λ the precession may be in the same direction as the orbital angular velocity, or opposite to it. Which is it for $\lambda > 0$?

(e) What is the value of n and the sign of λ for the correction due to the Sun's mass quadrupole moment, if that is caused by oblateness (the sun's polar axis being shorter than its equatorial diameter)?

a) $V(r) = -\frac{GmM}{r} - \lambda \frac{mMG}{r^n}$

Circular orbit:

$$-m\omega^2 R = -\nabla V$$

$$= -\frac{GMm}{r^2} - \frac{\lambda GMm}{r^{n+1}} n$$

$$m\omega_{orb}^2 R = GMm\left(\frac{1}{R^2} + \frac{\lambda n}{R^{n+1}}\right)$$

$$\omega^2 = GM\left(\frac{1}{R^3} + \frac{\lambda n}{R^{n+2}}\right)$$

b) $\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{1}{2}m r^2 \dot{\theta}^2}_{\frac{L^2}{2mr^2}} - V(r)$

$\dot\theta^2 = \omega^2$

$L = m\omega r^2$

$$V_{eff} = V(r) + \frac{L^2}{2mr^2} = -\frac{GmM}{r} - \lambda\frac{mMG}{r^n} + \frac{L^2}{2mr^2}$$

$$m\ddot{r} = -\partial_r V_{eff} = +\frac{L^2}{mr^3} - \frac{GMm}{r^2} - \frac{GMm\lambda n}{r^{n+1}}$$

$$\ddot{r} = \frac{L^2}{m^2 r^3} - GM\left(\frac{1}{r^2} + \frac{\lambda n}{r^{n+1}}\right)$$

let $r = R + \delta r$ $\quad \frac{1}{r^x} = \frac{1}{R^x\left(1+\frac{\delta r}{R}\right)^x} \approx \frac{1}{R^x}\left(1 - \frac{x\,\delta r}{R}\right)$

so $\delta\ddot{r} = \frac{L^2}{m^2R^3}\left(\cancel{1} - \frac{3\delta r}{R}\right) - GM\left(\frac{1}{R^2}\left(\cancel{1} - \frac{2\delta r}{R}\right) + \frac{\lambda n}{R^{n+1}}\left(\cancel{1} - \frac{(n+1)\delta r}{R}\right)\right)$

Red Terms Cancel using $L = m\omega_{orb}R$. In other words, to 1st order, no acceleration

It's circular

$$\delta\ddot{r} = \underbrace{\left(-\frac{3L^2}{m^2R^4} - GM\left(-\frac{2}{R^3} - \frac{\lambda n(n+1)}{R^{n+2}}\right)\right)}_{-\omega_{osc}^2}\delta r$$

$L = m\omega_{orb}R^2$

$-\omega_{osc}^2$ $\qquad L = m\omega_{orb}R^2$

c) $\omega_{osc}^2 = \frac{3L^2}{m^2R^4} - GM\left(\frac{2}{R^3} + \frac{\lambda n(n+1)}{R^{n+1}}\right) = 3\omega_{orb}^2 - \underbrace{GM\left(\frac{2}{R^3}\right.}+\left.\frac{\lambda n(n+1)}{R^{n+1}}\right)$

$\omega_{orb}^2 = GM\left(\frac{1}{R^3} + \frac{\lambda n}{R^{n+2}}\right)$

$= 3\omega_{orb}^2 - \left(2\omega_{orb}^2 - \frac{2GM\lambda n}{R^{n+2}}\right) - \frac{\lambda n(n+1)}{R^{n+1}}GM$

$= \omega_{orb}^2 + GM\left(\frac{2\lambda n}{R^{n+2}} - \frac{\lambda n(n+1)}{R^{n+1}}\right)$

$+ \frac{GM\lambda n}{R^{n+2}}\left(2 - (n+1)R\right)$

d) extra term is negative , so oscillation is slightly slower than orbit.

it precesses In the direction of orbital velocity, as observed