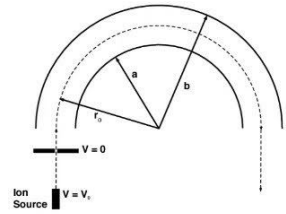
3. This problem explores some elements of a mass spectrometer. Parts (a) and (b) may be answered independently and treated non-relativistically.

(a) An ion of charge $+q$ and mass m is accelerated through a potential V_o as shown in Fig. 3(A). It enters a region between two very long (in the direction perpendicular to the page) cylindrical electrodes of radius a and b respectively. Find the potentials $V(a)$ and $V(b)$ such that the ion moves in a circle of radius r_o.



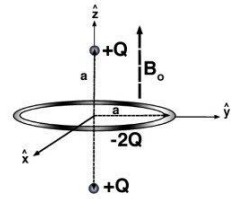


Figure 3: (A) Electrostatic filter. (B) Penning trap.

(b) A Penning trap is used in Fourier-transform mass spectrometry. At the simplest level, the Penning trap consists of a uniform magnetic field $B_o\hat{z}$ and a quadrupole electric field. Let us assume that the electric field is generated by two positive charges $+Q > 0$ located at $\pm a\hat{z}$ and a uniformly charged ring of radius a and charge $-2Q$ centered around the origin in the xy-plane, as shown in Fig. 3(B). This setup is rotationally symmetric around the z-axis.

i. Close to the origin, the electric field takes the form

$$\mathbf{E} = k_z z\hat{z} + k_r r\hat{r}\,,$$

where $r = \sqrt{x^2+y^2}$. Determine k_z and k_r.

While the general motion of an ion of mass m and charge $q > 0$ in the Penning trap is quite complicated, here we investigate only two particular cases:

ii. Find the frequency ω_z of small oscillations around the origin in the case where the ion moves only along the z-axis.

iii. Assume the ion moves *uniformly* along a circle of radius $R \ll a$ in the plane at $z = 0$. What is the angular frequency for this motion? Interpret the answer in the limit of large B_o.

$qV_0 = \frac{1}{2}mv^2 \rightarrow v = \sqrt{2\frac{q}{m}V_0}$

$m\frac{v^2}{r_0} = qE_r$ $\quad E = -\nabla V$ $\quad E_r = -\partial_r V$

$\frac{m}{r_0}2\frac{q}{m}V_0 = -q\partial_r V$

$\frac{2V_0}{r_0} = -\partial_r V$ or $r_0 = -\frac{2V_0}{\partial_r V}$

$\nabla^2 V = 0$ inside Tube

$V(r,\theta) = A + B\ln r + \sum_\ell (C_\ell r^\ell + D_\ell r^{-\ell})(E_\ell \cos\ell\theta + F_\ell \sin\ell\theta)$

$E_\ell = F_\ell = 0$ No θ dependence

$V(r) = A + B\ln r$

$V(a) = V_a = A + B\ln a$ } subtract $V_A - V_B = B\ln\frac{a}{b}$

$V(b) = V_b = A + B\ln b$ $\quad B = \frac{V_A - V_B}{\ln a/b}$

$V_b = A + (V_A - V_B)\frac{\ln b}{\ln a/b}$

$A = V_b - (V_A - V_B)\frac{\ln b}{\ln a/b}$

$V(r) = V_b - (V_A - V_b)\frac{\ln b}{\ln \frac{a}{b}} + \frac{V_A - V_b}{\ln(\frac{a}{b})}\ln(r)$

$= V_b + \frac{V_A - V_b}{\ln a/b}(-\ln b + \ln r) = V_b + \frac{V_A - V_B}{\ln a/b}\ln r/b$

OR $= V_A + \frac{V_B - V_A}{\ln b/a}\ln r/a$

$\partial_r V(r) = \frac{V_B - V_A}{\ln b/a}\frac{a}{r}\left(\frac{1}{a}\right) = \frac{V_B - V_A}{\ln b/a}\frac{1}{r}$

$V(r_0) = 0$ (Choice of Gauge Could be any Constant)

Mistake here in Andy's version

$0 = V_A + \frac{V_B - V_A}{\ln b/a}\ln r_0/a \Rightarrow -V_A \ln b/a = (V_B - V_A)\ln r_0/a \Rightarrow V_A\left(-\ln\frac{b}{a} + \ln\frac{r_0}{a}\right) = V_B$

$V_A \frac{\ln r_0/b}{\ln r_0/a} = V_B$

$$r_0 = \frac{-2V_0 \ln b/a \; r_0}{(V_B - V_A)} \longrightarrow 1 = \frac{-2V_0 \ln b/a}{V_B - V_A}$$

$$V_A\left(\frac{\ln r_0/b}{\ln r_0/a}\right) - V_A = -2V_0 \ln b/a$$

$$\frac{\ln r_0/b - \ln r_0/a}{\ln r_0/a} = -\frac{2V_0 \ln b/a}{V_A} \Longrightarrow \frac{1}{\ln r_0/a} = \frac{2V_0}{V_A}$$

$$\frac{\ln a/b}{\ln r_0/a} = \frac{-\ln b/a}{\ln r_0/a}$$

$$\boxed{V_A = 2V_0 \ln b/a}$$

CHECK $0 = 1 + \frac{\ln r_0/b - 1}{\ln r_0/a} \ln b/a$

$1 + \frac{\ln r_0/b - \ln r_0/a}{\ln b/a} = 1 + \frac{\ln a/b}{\ln b/a} = 0$

$$V_0 = 2V_0 \ln r/a \left(\frac{\ln r_0/b}{\ln r_0/a}\right) = V_b = 2V_0 \ln r_0/b$$

(b) A Penning trap is used in Fourier-transform mass spectrometry. At the simplest level, the Penning trap consists of a uniform magnetic field $B_o\hat{z}$ and a quadrupole electric field. Let us assume that the electric field is generated by two positive charges $+Q > 0$ located at $\pm a\hat{z}$ and a uniformly charged ring of radius a and charge $-2Q$ centered around the origin in the xy-plane, as shown in Fig. 3(B). This setup is rotationally symmetric around the z-axis.

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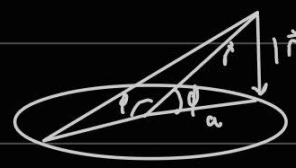
b) E from the two charges acts in the $\hat{z}$ direction, E from the ring acts in $\hat{r}$ (cylindrical) direction

$$V(\vec{r}) = \frac{kQ}{|\vec{r} - a\hat{z}|} + \frac{kQ}{|\vec{r} + a\hat{z}|} + V_{ring}$$

$$V_{ring} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r')\,d^3r'}{(\vec{r} - r')} = \int_0^{2\pi} \frac{\lambda a\, d\theta}{|\vec{r} - a\hat{r}|} = \int_0^{2\pi} \frac{\lambda a\, d\theta}{\sqrt{|\vec{r}|^2 + a^2 - 2a|\vec{r}|\cos\phi}}$$

$z^2 + r^2 \leftarrow$ polar



$$= \int_0^{2\pi} \frac{\lambda a\, d\theta}{\sqrt{|\vec{r}|^2 + a^2 - 2a|\vec{r}|\cos\theta}} = \int_0^{2\pi} \frac{\lambda a\, d\theta}{a\sqrt{\frac{|\vec{r}|^2}{a^2} + 1 - \frac{2|\vec{r}|}{a}\cos\theta}}$$

Assume $\phi = \theta$ near the origin ($r \ll a$) ϕ should depend on θ

$$\approx \lambda \int_0^{2\pi} d\theta \left(1 - \frac{1}{2}\left(\frac{r^2}{a^2} - \frac{2r}{a}\cos\theta\right) + \overbrace{\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}}^{3/8}\left(\frac{r^2}{a^2} - \frac{2r}{a}\cos\theta\right)^2\right)$$

Integrates to 0

Only term order $\frac{r^2}{a^2}$ is $\frac{4r^2}{a^2}\cos^2\theta$

$$= \lambda \int_0^{2\pi} d\theta \left(1 - \frac{r^2}{2a^2} + \frac{3}{4}\frac{r^2}{a^2}\cos^2\theta\right)$$

$$\int_0^{2\pi} \cos^2 x = \pi$$

$$= \lambda\left(2\pi - \frac{\pi(z^2+r^2)}{a^2} + \frac{3\pi}{4}\frac{(z^2+r^2)}{a^2}\right)$$ ← Sub back in from $|\vec{r}|$

$$\lambda = \frac{-2Q}{2\pi a} = -\frac{Q}{\pi a}$$

$$= 2\pi\lambda + \lambda\frac{(z^2+r^2)}{a^2}\left(-\pi + \frac{3\pi}{4}\right) = \frac{2\pi\lambda - \frac{\pi}{4}\lambda\frac{(z^2+r^2)}{a^2}}{4\pi\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\left(\frac{-2Q}{a} + \frac{Q(z^2+r^2)}{4a^3}\right)$$

$$V(r,z) = \frac{Q}{4\pi\varepsilon_0}\left(\frac{|\vec{r}-a\hat{z}| + |\vec{r}+a\hat{z}|}{|\vec{r}-a\hat{z}|\cdot|\vec{r}+a\hat{z}|}\right) \quad (2\vec{r})$$

Charges

$$\sqrt{(z-a)^2+r^2} \qquad \sqrt{(z+a)^2+r^2}$$

$$\sqrt{a^2\left(\frac{z}{a}-1\right)^2 + r^2} \qquad \sqrt{a^2\left(\frac{z}{a}+1\right)^2 + r^2}$$

$$a\sqrt{1 - \frac{2z}{a} + \frac{r^2+z^2}{a^2}} \qquad a\sqrt{1 + \frac{2z}{a} + \frac{r^2+z^2}{a^2}}$$

$$a^2\left[\left(1 + \frac{2z}{a} + \frac{r^2+z^2}{a^2} - 2\frac{z}{a}\left(1 + \frac{2z}{a} + \frac{r^2+z^2}{a^2}\right) + \frac{r^2+z^2}{a^2}\left(1 + \frac{2z}{a} + \frac{r^2+z^2}{a^2}\right)\right)\right]^{1/2}$$

small small small

Ignore z^3/a^3

$$a^2\left(1 + 2\frac{r^2+z^2}{a^2} - \frac{4z^2}{a^2}\right) = a^2\left(1 + \frac{2(r^2-z^2)}{a^2}\right)^{1/2}$$

$$V_{Charges} = \frac{Q}{4\pi\varepsilon_0}\frac{2|r|}{a^2\left(1 + \frac{2(r^2-z^2)}{a^2}\right)^{1/2}} \approx \frac{Q}{2\pi\varepsilon_0}\frac{|r|}{a^2}\left(1 - \frac{r^2-z^2}{a^2}\right) \approx \frac{Q}{2\pi\varepsilon_0 a^2}\left(1 - \frac{r^2-z^2}{a^2}\right)(z^2+r^2)^{1/2}$$

$1 - \frac{r^2}{a^2} + \frac{z^2}{a^2}$

$$V(r,z) = \frac{Q}{2\pi\varepsilon_0 a^2}\left(1 - \frac{r^2-z^2}{a^2}\right)(z^2+r^2)^{1/2} + \frac{1}{4\pi\varepsilon_0}\left(\frac{-2Q}{a} + \frac{Q(z^2+r^2)}{4a^3}\right)$$

$$E_r = -\partial_r V = \frac{Q}{2\pi\varepsilon_0 a^2}\left[\left(-\frac{2r}{a^2}\right)(z^2+r^2) + \left(1 - \frac{r^2-z^2}{a^2}\right)r(z^2+r^2)^{-1/2}\right] + \frac{1}{4\pi\varepsilon_0}\frac{2Qr}{4a^3}$$

$$= r\left[\frac{-Q}{2\pi\varepsilon_0 a^2}\left[-\frac{2}{a^2}(z^2+r^2) + \left(1 - \frac{r^2-z^2}{a^2}\right)(z^2+r^2)^{-1/2}\right] - \frac{1}{4\pi\varepsilon_0}\frac{2Q}{4a^3}\right]$$

K_r

$$E_z = -\partial_z V = \frac{-Q}{2\pi\varepsilon_0 a^2}\left(\frac{2z}{a^2}(z^2+r^2)^{1/2} + \left(1 - \frac{r^2-z^2}{a^2}\right)z(z^2+r^2)^{-1/2}\right) + \frac{-Q}{4\pi\varepsilon_0}\frac{2z}{4a^3}$$

$$= z\left[\frac{-Q}{2\pi\varepsilon_0 a^2}\left(\frac{2}{a^2}(z^2+r^2)^{1/2} + \left(1 - \frac{r^2-z^2}{a^2}\right)(z^2+r^2)^{-1/2}\right) - \frac{2Q}{4\pi\varepsilon_0 4a^3}\right]$$

K_z

ii) $m\ddot{z} = qE_z = qk_z z$

↑ ion ↑ negative number

$\omega_z^2 = qk_z$ so $\omega_z = \sqrt{qk_z}$

iii) $F_r = 0 = m\omega^2 r + qk_r r - q\omega r B$

$m\omega^2 + qk_r - q\omega B$

$\omega^2 - \underbrace{\frac{q}{m}B}_{\omega_L}\omega + qk_r = \omega^2\left(1 - \frac{\omega_L}{\omega} + \frac{qk_r}{\omega^2}\right)$

ω_L Larmor frequency

↑ $\omega \gg \omega_L$ in Large B limit

So $\omega^2\left(1 + \frac{qk_r}{\omega^2}\right) = 0$

$\omega^2 = qk_r \Rightarrow \omega_r = \sqrt{qk_r}$