J15E2 (Solution by Jim Wu)
A Fresnel rhomb is an optical device used to convert linearly polarized light into circularly polarized light. As shown in Fig 2, light hits the surface of the rhomb at normal incidence, it then undergoes two total internal reflections inside the rhomb, and then leaves the rhomb again normally.
The total internal reflections are such that each reflection generates a phase difference of 45° between the components of the light wave that is parallel and the component that is perpendicular to the plane of incidence (the plane of the page in Fig. 2), and so after two internal reflections a lightwave that was originally linearly polarized at 45° with respect to the plane of incidence becomes circularly polarized.

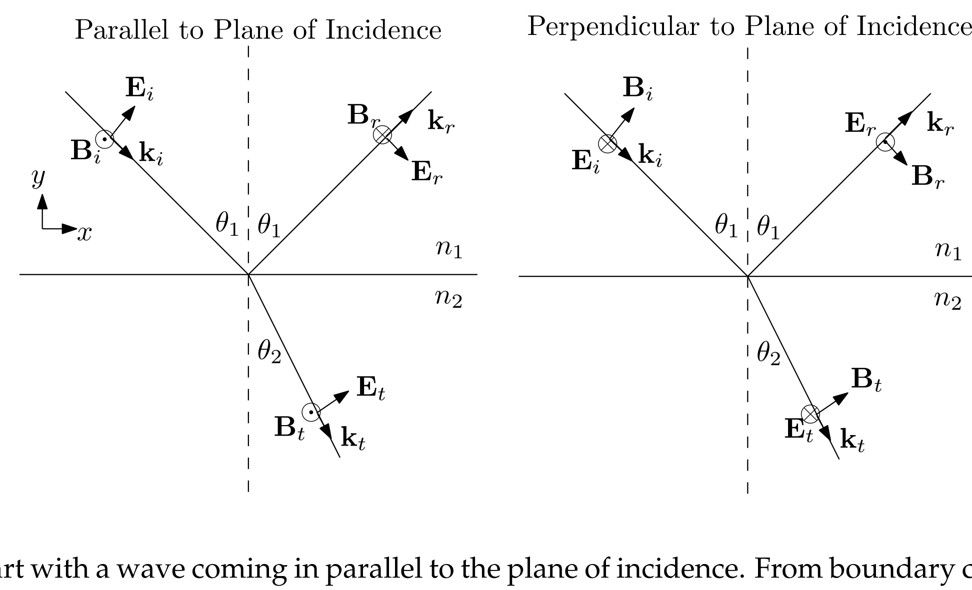
(a) For a single internal reflection, find the phase shift that the reflected wave acquires relative to the incident wave assuming the electromagnetic wave is polarized in the plane of incidence.

(b) Calculate the phase shift that the reflected wave acquires relative to the incident wave when the electromagnetic wave is polarized perpendicular to the plane of incidence.

(c) If each of the two total internal reflections in a Fresnel rhomb occurs at an angle of incidence of $\theta_i = 53.3°$, calculate the index of refraction n of the Fresnel rhomb relative to that of the surrounding medium.

Solution:

(a) In order to do this problem, we need the Fresnel formulas, which describe the reflected and transmitted EM wave amplitudes relative that of the incident wave with polarization parallel or perpendicular to the plane of incidence. I certainly do not remember the formulas and would need to re-derive them from scratch by using the boundary conditions for EM waves.



Let's start with a wave coming in parallel to the plane of incidence. From boundary conditions derived from Maxwell's equations,

$$\begin{aligned}
\mathbf{n}\cdot\mathbf{D}=0 \quad &\Rightarrow \quad \epsilon(E_i-E_r)\sin\theta_1=\epsilon_0 E_t\sin\theta_2\\
\mathbf{n}\times\mathbf{E}=0 \quad &\Rightarrow \quad (E_i+E_r)\cos\theta_1=E_t\cos\theta_2\\
\mathbf{n}\cdot\mathbf{B}=0 \quad &\Rightarrow \quad 0=0\\
\mathbf{n}\times\mathbf{H}=0 \quad &\Rightarrow \quad \frac{n_1}{\mu_1 c}(E_i-E_r)=\frac{n_2}{\mu_2 c}E_t
\end{aligned}$$

Note that we used the fact that $B=E/c$ and $v=c/n$ in a linear medium. Using the second and fourth relation and assuming that $\mu_1\approx\mu_2$, we have

$$\begin{aligned}
(E_i-E_r)\cos\theta_1 &= \frac{n_1}{n_2}(E_i+E_r)\cos\theta_2\\
\left(\frac{E_i}{E_r}\right)_{\parallel} &= \frac{n_1\cos\theta_2-n_2\cos\theta_1}{n_1\cos\theta_2+n_2\cos\theta_1}
\end{aligned}$$

If we repeat the same calculations with a wave perpendicular to the plane of incidence, we have boundary conditions

$$\begin{aligned}
\mathbf{n}\cdot\mathbf{D}=0 \quad &\Rightarrow \quad 0=0\\
\mathbf{n}\times\mathbf{E}=0 \quad &\Rightarrow \quad E_i-E_r=E_t\\
\mathbf{n}\cdot\mathbf{B}=0 \quad &\Rightarrow \quad \frac{n_1}{c}(E_i-E_r)\sin\theta_1=\frac{n_2}{c}E_t\sin\theta_2\\
\mathbf{n}\times\mathbf{H}=0 \quad &\Rightarrow \quad \frac{n_1}{\mu_1 c}(E_i+E_r)\cos\theta_1=\frac{n_2}{\mu_2 c}E_t\cos\theta_2
\end{aligned}$$

Combining the second and fourth equation and employing the same approximations, we have

$$\begin{aligned}
n_1(E_i-E_r)\cos\theta_1 &= n_2(E_i+E_r)\cos\theta_2\\
\left(\frac{E_r}{E_i}\right)_{\perp} &= \frac{n_1\cos\theta_1-n_2\cos\theta_2}{n_1\cos\theta_1+n_2\cos\theta_2}
\end{aligned}$$

From Snell's law, $n_1\sin\theta_1=n_2\sin\theta_2$. Let's call the Fresnel rhomb medium 1 with index of refraction $n_1=n$ and the air medium 2 with index of refraction of $n_2\approx 1$. Then for internal reflection, we have

$$n^2\sin^2\theta_1=1-\cos^2\theta_2 \qquad \Rightarrow \qquad \cos\theta_2=\sqrt{1-n^2\sin^2\theta_1}=in\sqrt{\sin^2\theta_1-\frac{1}{n^2}}$$

Note that for internal reflection, $n\sin\theta_1>1$ and so $\cos\theta_2$ is imaginary. Substituting our results back into $(E_r/E_i)_{\parallel}$, the reflection coefficient is

$$r_{\parallel}=\frac{in^2\sqrt{\sin^2\theta_i-\frac{1}{n^2}}-\cos\theta_i}{in^2\sqrt{\sin^2\theta_i-\frac{1}{n^2}}+\cos\theta_i}=\frac{-n^4\left(\sin^2\theta_i-\frac{1}{n^2}\right)+\cos^2\theta_i-2in^2\cos\theta_i\sqrt{\sin^2\theta_i-\frac{1}{n^2}}}{n^4\left(\sin^2\theta_i-\frac{1}{n^2}\right)+\cos^2\theta_i}$$

Hence, the phase between the incident and reflected wave for parallel incidence is given by

$$\tan\phi_{\parallel}=\frac{-2n^2\cos\theta_i\sqrt{\sin^2\theta_i-\frac{1}{n^2}}}{n^2-n^4\sin^2\theta_i+\cos^2\theta_i}$$

(b) Similarly, for the perpendicular case,

$$r_\perp = \frac{n\cos\theta_i - in\sqrt{\sin^2\theta_i - \frac{1}{n^2}}}{n\cos\theta_i + in\sqrt{\sin^2\theta_i - \frac{1}{n^2}}} = \frac{\cos^2\theta_i - \sin^2\theta_i + \frac{1}{n^2} - 2i\cos\theta_i\sqrt{\sin^2\theta_i - \frac{1}{n^2}}}{\cos^2\theta_i + \sin^2\theta_i - \frac{1}{n^2}}$$

and the phase is given by

$$\tan\phi_\perp = \frac{-2\cos\theta_i\sqrt{\sin^2\theta_i - \frac{1}{n^2}}}{\cos^2\theta_i - \sin^2\theta_i + \frac{1}{n^2}}$$

(c) Notice that both reflection coefficients are of the form

$$\frac{a - ib}{a + ib} = \frac{a^2 - b^2 - 2iab}{a^2 + b^2}$$

which means that the phase is

$$\tan\phi = -\frac{2ab}{a^2 - b^2} = \frac{2(a/b)}{1 - (a/b)^2}$$

Recall that

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$$

and so

$$\tan 2\left(\frac{\phi}{2}\right) = \frac{2\tan\frac{\phi}{2}}{1 - \tan^2\frac{\phi}{2}}$$

which suggests that

$$\tan\frac{\phi}{2} = \frac{a}{b}$$

Now defining $\Delta\phi = \phi_\perp - \phi_\parallel$ and using the tangent of a difference of angles, we have

$$\begin{aligned}
\tan\left(\frac{\Delta\phi}{2}\right) &= \frac{\tan\frac{\phi_\perp}{2} - \tan\frac{\phi_\parallel}{2}}{1 + \tan\frac{\phi_\parallel}{2}\tan\frac{\phi_\perp}{2}} \\
&= \frac{\left(\frac{\cos\theta_i}{\sqrt{\sin^2\theta_i - \frac{1}{n^2}}}\right) - \left(\frac{\cos\theta_i}{n^2\sqrt{\sin^2\theta_i - \frac{1}{n^2}}}\right)}{1 + \left(\frac{\cos\theta_i}{n^2\sqrt{\sin^2\theta_i - \frac{1}{n^2}}}\right)\left(\frac{\cos\theta_i}{\sqrt{\sin^2\theta_i - \frac{1}{n^2}}}\right)} \\
&= \frac{(n^2 - 1)\cos\theta_i\sqrt{\sin^2\theta_i - \frac{1}{n^2}}}{n^2\sin^2\theta_i - 1 + \cos^2\theta_i} \\
&= \frac{\cos\theta_i\sqrt{\sin^2\theta_i - \frac{1}{n^2}}}{\sin^2\theta_i}
\end{aligned}$$

Solving for n, we have

$$\sqrt{\sin^2\theta_i - \frac{1}{n^2}} = \tan\left(\frac{\Delta\phi}{2}\right)\tan\theta_i\sin\theta_i$$
$$n = \frac{1}{\sin\theta_i\sqrt{1-\tan^2\frac{\Delta\phi}{2}\tan^2\theta_i}}$$

Note that we take the positive root since n is a positive number. For $\Delta\phi = 45^\circ$ and $\theta_i = 53.3^\circ$, the index of refraction of the Fresnel rhomb must be

$$n = \frac{1}{\sin(53.3^\circ)\sqrt{1-\tan^2(22.5^\circ)\tan^2(53.3^\circ)}} \approx 1.5$$

which is around the index of refraction for glass.

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