J15E1 (Solution by Jim Wu)
A small wire loop of radius a lies in the xy-plane, centered on the origin. A magnetic moment $\mathbf{m} = m\hat{z}$ travels up along the z axis with constant speed v. It passes through the center of the wire loop at $t = 0$.

(a) Compute the emf $\varepsilon(t)$ around the loop.

(b) If the loop has resistance R, find the Joule heat $P(t)$. Assume the loop is fixed in position.

(c) Now consider the case where a uniform linear charge density λ is glued to a non-conducting loop (same orientation and radius as above), and the loop is allowed to spin. What is the position of $\mathbf{m}$ at the time the loop attains its largest angular momentum, $\mathbf{L}_{max}$? What is the value of $\mathbf{L}_{max}$? Assume the dipole began its constant-velocity motion at $t = -\infty$, and the loop was at rest then.

Solution:

(a) To compute the emf, we need to first calculate the magnetic flux through the loop. Recall that the magnetic field due to a point dipole is given by

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right) \\
&= \frac{\mu_0}{4\pi}\left(\frac{3m\cos\theta\hat{\mathbf{r}} - m\hat{\mathbf{z}}}{r^3}\right) \\
&= \frac{\mu_0}{4\pi}\left(\frac{3m\cos\theta\hat{\mathbf{r}} - m(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}})}{r^3}\right) \\
&= \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})
\end{aligned}$$

Note that we are assuming the origin is at the center of the dipole. Placing the dipole at the origin and working in the dipole reference frame will make the computation of Φ_B much simpler.

Let's say at a moment t, the dipole is a height z below the ring. Then, the flux at this instance in time, using a spherical cap surface of radius $r = \sqrt{z^2 + a^2}$ for integration, is

$$\begin{aligned}
\Phi_B = \int \mathbf{B}\cdot d\mathbf{A} &= \int_0^{\sin^{-1}\frac{a}{r}} \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}})\cdot 2\pi r^2\sin\theta\, d\theta\hat{\mathbf{r}} \\
&= \frac{\mu_0 m}{r}\int_0^{\sin^{-1}\frac{a}{r}} \cos\theta\sin\theta\, d\theta \\
&= \frac{\mu_0 m}{2r}\left[\sin^2\left(\sin^{-1}\frac{a}{r}\right) - \sin^2(0)\right] \\
&= \frac{\mu_0 m a^2}{2r^3}
\end{aligned}$$

Note that in the frame of the magnetic dipole, the ring's position is $z = -vt$ and $r = \sqrt{z^2 + a^2}$, then

$$\Phi_B = \frac{\mu_0 a^2}{2(v^2t^2 + a^2)^{3/2}}$$

Therefore, the emf in the conducting loop is

$$\varepsilon(t) = \left|\frac{d\Phi_B}{dt}\right| = \left|-\frac{3\mu_0 m a^2 v^2 t}{4(v^2t^2 + a^2)^{5/2}}\right| = \frac{3\mu_0 m a^2 v^2 t}{4(v^2t^2 + a^2)^{5/2}}$$

(b) The power dissipated by the loop is

$$P(t) = \frac{\varepsilon(t)^2}{R} = \frac{9\mu_0^2 m^2 a^4 v^4 t^2}{16(v^2t^2 + a^2)^5}$$

(c) A change magnetic flux induces an electric field in the $\hat{\phi}$ direction that acts on the charged non-conducting loop. For $t < 0$, as the magnetic dipole gets closer to loop, the induced electric field must point in the $-\hat{\boldsymbol{\phi}}$ direction as viewed from above. For $t > 0$, when the dipole moves aways from the loop, the electric field will flip directions and point in the $+\hat{\boldsymbol{\phi}}$ direction. So,

$$2\pi a E_\phi = -\frac{d\Phi_B}{dt} \qquad \Rightarrow \qquad \mathbf{E} = \frac{3\mu_0 m a v^2 t}{8\pi(v^2t^2 + a^2)^{5/2}}\hat{\boldsymbol{\phi}}$$

This agrees with out intuition since at $t < 0$, $\mathbf{E}$ is in the $-\hat{\boldsymbol{\phi}}$ and at $t > 0$, the electric field $\mathbf{E}$ is in the $\hat{\boldsymbol{\phi}}$ direction.

The torque on the ring is

$$\boldsymbol{\tau} = \mathbf{r} \times \lambda(2\pi a)\mathbf{E} = \frac{3\mu_0 \lambda m a^3 v^2 t}{4(v^2t^2 + a^2)^{5/2}}\hat{\mathbf{z}}$$

and so, the angular momentum is maximum when the torque is zero, or when $t = 0$. The angular momentum as a function of time is

$$\begin{aligned}
\mathbf{L} = \int \boldsymbol{\tau}\, dt &= \frac{3\mu_0 \lambda m a^3 v^2}{4} \int_{-\infty}^{t} \frac{t}{(v^2t^2 + a^2)^{5/2}}\hat{\mathbf{z}} \\
&= \frac{3\mu_0 \lambda m a^3 v^2}{4}\left[-\frac{1}{3v^2(v^2t^2 + a^2)^{3/2}}\right]_{-\infty}^{t}\hat{\mathbf{z}} \\
&= -\frac{\mu_0 \lambda m a^3}{4(v^2t^2 + a^2)^{3/2}}\hat{\mathbf{z}}
\end{aligned}$$

and the maximum angular momentum is at $t = 0$:

$$\mathbf{L}_{max} = -\frac{\mu_0 \lambda m}{4}\hat{\mathbf{z}}$$

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