Prelims Solutions

Problem J14T2

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1

For a Fermi system with N particles in d dimensions at 0 temperature, all the states in the positive $|\vec{n}|$ sector are filled up to a certain radius $|\vec{n}|_{max} = n_F$ so $N \propto {n_F}^d$. We are given $E \propto |\vec{p}|^\nu \propto |\vec{n}|^\nu \to n_F \propto E^{\frac{1}{\nu}}$. Hence, $N \propto E^{\frac{d}{\nu}} \to E \propto N^{\frac{\nu}{d}} \to \lambda = \frac{\nu}{d}$.

2

The first correction to the energy from the Sommerfeld expansion for low temperature goes like $g'(\mu)T^2$ for integrals of the form $\int_0^\infty \frac{g(E)dE}{e^{\beta(E-\mu)}+1}$. For energy $g(E) = D(E)E = \frac{dN}{dE}E \propto E^{\frac{1}{\lambda}-1}E = E^\lambda$. The heat capacity $C_v = \frac{\partial U}{\partial T} \propto g'(\mu)T \propto E^{\frac{1}{\lambda}-1}T \propto N^{1-\lambda}T$.

3

$P = -\frac{\partial F}{\partial V} = \frac{k_B T}{Z}\frac{\partial Z}{\partial V}$ where $Z = \sum_i e^{-\beta E_i}$. For a gas, the energies of microstates in the Boltzmann factors of Z are functions of volume $E_i = A_i L^{-\nu} = A_i V^{-\frac{\nu}{d}}$ (where A_i is independent of volume) $\to \frac{\partial E_i}{\partial V} = -\frac{\nu}{d}\frac{E_i}{V}$. Since $E = \frac{1}{Z}\sum_i E_i e^{-\beta E_i}$, we get $P = \frac{\nu}{d}\frac{E}{V} \to \alpha = \frac{\nu}{d}$.

4

Our formula above predicts $\alpha = \frac{1}{3}$. Rough argument (see Feynman Lectures Vol. 1 for cleaner argument): Consider a wall in the zy plane with area A that is a part of container with N photons and volume V. In a time interval dt, only photons moving toward the wall will hit it and bounce off, imparting momentum $2p$ onto the wall. This is a number $\frac{1}{6}\frac{N}{V}Acdt$ photons where the 1/6 accounts for the fraction of photons moving in the direction of the wall. Total pressure from photons of momentum p and energy $E = pc$ is thus $P = \frac{momentum imparted}{Area} = \frac{\frac{1}{6}\frac{N}{V}Acdt * 2p}{Adt} = \frac{Npc}{3V} = \frac{E}{3V}$.