Prelims Solutions

Problem J14M2

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In the figure Ω is in the zy plane so we have symmetry of the ring in the x direction (unclear what α is for...). Consider an azimuthal angle ϕ on the ring measured using Ω as the "z axis" and $\phi = 0$ corresponding to the point on the ring farthest from the Sun. A small angle on the ring has mass $\frac{\delta M d\phi}{2\pi}$, has coordinates $(x, y, z) = R_r(sin(\phi), -cos(\phi)cos(\theta), cos(\phi)sin(\theta))$. The torque is thus:

$$\vec{\tau} = \int_0^{2\pi} \vec{r} \times \vec{F_g} = \int_0^{2\pi} \frac{G\frac{\delta M d\phi}{2\pi}M_s}{((R_r sin(\phi))^2 + (R_{es} + R_r cos(\phi)cos(\theta))^2 + (R_r cos(\phi)sin(\theta))^2)^{\frac{3}{2}}}$$

$$*(R_r sin(\phi), -R_r cos(\phi)cos(\theta), R_r cos(\phi)sin(\theta)) \times (-R_r sin(\phi), R_{es} + R_r cos(\phi)cos(\theta), -R_r cos(\phi)sin(\theta))$$

By symmetry, $\tau_y = 0$ and $\tau_z = 0$. The x component of the cross product gives $(-R_r cos(\phi)sin(\theta))(-R_r cos(\phi)cos(\theta)) - (R_{es} + R_r cos(\phi)cos(\theta))(R_r cos(\phi)sin(\theta)) \approx -R_{es}R_r cos(\phi)sin(\theta)$ to first order in R_r. Approximating the distance from the Sun using only the y coordinate gives:

$$\tau = -\int_0^{2\pi} \frac{G\frac{\delta M}{2\pi}M_s R_{es}R_r cos(\phi)sin(\theta)}{(R_{es} + R_r cos(\phi)cos(\theta))^3}d\phi = -\int_0^{2\pi} \frac{G\frac{\delta M}{2\pi}M_s R_{es}R_r cos(\phi)sin(\theta)}{R_{es}^3(1 + \frac{R_r}{R_{es}}cos(\phi)cos(\theta))^3}d\phi$$

$$\approx -\frac{G\frac{\delta M}{2\pi}M_s R_r sin(\theta)}{R_{es}^2}\int_0^{2\pi} \cos(\phi)(1 - 3\frac{R_r}{R_{es}}cos(\phi)cos(\theta))d\phi = -\frac{G\frac{\delta M}{2\pi}M_s R_r sin(\theta)}{R_{es}^2}(0 - 3\frac{R_r}{R_{es}}cos(\theta)(\pi))$$

Hence,

$$\tau_x = \frac{3G\delta M M_s R_r^2 sin(\theta)cos(\theta)}{2R_{es}^3}$$

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To first order in $\frac{\delta M}{M}$, $L = \frac{2}{5}MR_o^2\Omega$.

$$\tau_x = (\frac{d\vec{L}}{dt})_x = \frac{Lsin(\theta)d\theta}{dt} = Lsin(\theta)\omega_p \rightarrow$$

$$\omega_p = \frac{\tau_x}{Lsin(\theta)} = \frac{15G\delta M M_s R_r^2 cos(\theta)}{4M\Omega R_o^2 R_{es}^3}$$

i.e. counterclockwise around z. This is the instantaneous precession I think because the torque is probably going to change as the ring/earth moves in its orbit...