2. Consider a Fermi gas of N non-interacting particles in d dimensions where each particle has kinetic energy K.E. $= a|\vec{p}|^\nu$. The Fermi gas is placed in a box of volume V. Here, a and ν are positive constants, and N is assumed to be very large.

   (a) The Fermi energy can be written approximately as $E_F \approx \gamma N^\lambda$ for some γ and λ. Determine the exponent λ in terms of d and ν.

   (b) How does the heat capacity scale with temperature and the number of particles at small temperatures? Give the answer in terms of λ.

   (c) For this Fermi gas at temperature $T > 0$ the pressure P is related to the total energy E through $P = \alpha E/V$. Find α in terms of ν and d.

   <u>Hint:</u> P may be expressed through an appropriate derivative of the partition function. Think about how the energy of any given state changes with V.

   (d) For a relativistic Fermi gas in 3 dimensions $\nu = 1$. For this case derive $P = \alpha E/V$ also from the kinetic theory, with P expressed as the force per unit area exerted by the gas particles on the walls of the container.

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a) $\varepsilon = \alpha|\vec{p}|^{\nu} \leftarrow |\vec{p}| \equiv p$

$= \alpha p^{\nu} \leftarrow p = \hbar k,\ k = \frac{\pi}{L} n$

$= \alpha\left(\hbar \frac{\pi}{L}\right)^{\nu} n^{\nu}$

$\approx n^{\nu}$

$N \approx n_f^{d} \leftarrow n^{\nu} = \varepsilon$

$\approx \varepsilon_f^{d/\nu}$

$\varepsilon_f \approx N^{\nu/d}$

$$\boxed{\lambda = \nu/d}$$

b) $g(\varepsilon) = \frac{\partial N}{\partial \varepsilon} \approx \frac{d}{d\varepsilon}\left(\varepsilon^{1/\lambda}\right) \approx \varepsilon^{\frac{1}{\lambda}-1} = g_0\, \varepsilon^{\frac{1}{\lambda}-1} = g_0\, \varepsilon^{\frac{1-\lambda}{\lambda}} = g_0 N^{1-\lambda}$

$E = \int_0^{\infty} f(\varepsilon) g(\varepsilon)\, \varepsilon\, d\varepsilon = g_0 N^{1-\lambda} \int_0^{\infty} \frac{\varepsilon}{e^{\beta\varepsilon}+1} d\varepsilon \leftarrow x = \beta\varepsilon = \frac{\varepsilon}{k_B T}$

$= g_0 N^{1-\lambda} k_B^2 T^2 \int_0^{\infty} \frac{x}{e^{x}+1} dx \approx N^{1-\lambda} T^2$

$C = \frac{\partial E}{\partial T} \approx N^{1-\lambda} T$

$$\boxed{C \approx N^{1-\lambda} T}$$

c) $P = -\frac{\partial E}{\partial V} \leftarrow E = \alpha(\hbar\pi n)^{\nu} L^{-\nu} = \alpha(\hbar\pi n)^{\nu} V^{-\nu/d} = \alpha(\hbar\pi n)^{\nu} V^{-\lambda}$

$= -\left[\alpha(\hbar\pi n)^{\nu}\left(-\lambda V^{-\lambda-1}\right)\right] = \lambda\left[\left(\alpha(\hbar\pi n)^{\nu} V^{-\lambda}\right)/V\right] = \lambda \frac{E}{V}$

$$\boxed{\alpha = \lambda = \nu/d}$$

d) Expect $\alpha = \frac{1}{3} \leftarrow \nu = 1, d = 3$

$$P = \frac{F}{A} \leftarrow F = \frac{dp}{dt}$$

$$= \frac{dp}{A\,dt} = \frac{\text{change in momentum}}{\text{area} \times \text{time interval}} = \frac{\#\text{particles changing momentum}}{A\,dt}\,2p$$

$$\#_{\Delta p} = \frac{N}{V}(\text{Volume of a single collision}) \equiv \frac{N}{V}\,dV_c$$

Take a box of volume V, side area A, and side length L. Only $\frac{1}{6}$ of the total surface area of the box is involved

$$dV_c = \frac{1}{6}A\,dx \leftarrow dx = ct$$

$$= \frac{1}{6}Ac\,dt$$

$$\#_{\Delta p} = \frac{N}{V}\frac{1}{6}Ac\,dt$$

$$P = \left(\frac{N}{V}\frac{1}{6}Ac\,dt\right)2p\left(\frac{1}{A\,dt}\right) = \frac{N}{V}\frac{1}{3}pc = \frac{1}{3}\frac{Npc}{V} \leftarrow E = N\varepsilon_{particle} = Npc$$

$$= \frac{1}{3}\frac{E}{V}$$

$$\boxed{P = \alpha\frac{E}{V},\ \alpha = \frac{1}{3}}$$