9 Dec 2021

2. Scattering from a spherical potential:

   (a) Calculate the differential cross-section, $d\sigma/d\Omega$, for a particle of mass m scattering from a spherical potential $V(r) = V_0 e^{-(r/a)^2}$ using the first-order Born approximation. You may need the integral

   $$\int_0^\infty \sin r\, e^{-(r/b)^2} r dr = \frac{\sqrt{\pi}}{4} b^3 e^{-b^2/4}.$$

   (b) Calculate the total cross-section. It may be helpful to use the representation $|\vec{k} - \vec{k}'| = 2|\vec{k}| \sin(\theta/2)$, where θ is the angle between $\vec{k}$ and $\vec{k}'$.

   (c) What are the conditions on V_0, a and/or k for the first-order Born approximation to be valid?

J14Q.2

a) $V = V_0 e^{-(r/a)^2}$ ← spherically symmetric

$$f(\theta) = -\frac{2m}{\hbar^2 K}\int_0^\infty r V(r)\sin(Kr)\,dr \quad \leftarrow K = 2k\sin(\theta/2)$$

$$= -\frac{2mV_0}{\hbar^2 K^3}\int_0^\infty (Kr)\sin(Kr)\, e^{-r^2/a^2}\, d(Kr) \quad \leftarrow u = Kr,\ r^2 = u^2/K^2$$

$$= -\frac{2mV_0}{\hbar^2 K^3}\int_0^\infty u\sin(u)\, e^{-u^2/K^2a^2}\,du \quad \leftarrow \int_0^\infty \sin r\, e^{-(r/b)^2} r\,dr = \frac{\sqrt{\pi}}{4} b^3 e^{-b^2/4}. \ , b = aK$$

$$= -\frac{2mV_0}{\hbar^2 K^3}\left(\frac{\sqrt{\pi}}{4}K^3a^3 e^{-a^2K^2/4}\right) = -\frac{mV_0a^3}{2\hbar^2}\sqrt{\pi}\, e^{-a^2K^2/4}$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{m^2V_0^2\pi^2a^6}{4\hbar^4} e^{-a^2K^2/2}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{m^2V_0^2\pi^2a^6}{4\hbar^4} e^{-2a^2k^2\sin^2(\theta/2)}}$$

b) $\sigma = \int d\sigma = \int \frac{d\sigma}{d\Omega}d\Omega$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{m^2V_0^2\pi^2a^6}{4\hbar^4} e^{-a^2K^2/2}\sin\theta\, d\theta\, d\phi$$

$$= \frac{m^2V_0^2\pi^3a^6}{2\hbar^4}\int_0^{\pi} \sin\theta\, e^{-2a^2k^2\sin^2(\theta/2)}\, d\theta \quad \leftarrow u = 2a^2k^2\sin^2(\theta/2);\ du = 4k^2a^2\sin\left(\tfrac{\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right)d\theta$$

$$= 2k^2a^2\sin(\theta)\,d\theta$$

$$= \frac{m^2V_0^2\pi^3a^6}{2\hbar^4}\,\frac{1}{2k^2a^2}\int_0^{2a^2k^2} e^{-u}\,du$$

$$= \frac{m^2V_0^2\pi^3a^4}{4\hbar^4k^2}\left[1 - e^{-2a^2k^2}\right]$$

$$\boxed{\sigma = \frac{m^2V_0^2\pi^3a^4}{4\hbar^4k^2}\left[1 - e^{-2a^2k^2}\right]}$$

c) low energy potential: $\boxed{V_0 \ll \frac{\hbar^2k^2}{2m}}$

tightly localized: $\boxed{ak \gg 1}$