1. A particle of mass m, with the Hamiltonian $H = \frac{p^2}{2m} + V(x)$, is moving in one dimension subject to an attractive potential of the form:

$$V(x) \;=\; -U\left[\delta(x + a/2) + \delta(x - a/2)\right]$$

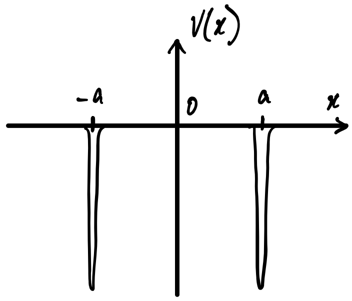
with $U > 0$.

(a) What consequences does the Hamiltonian's reflection symmetry have for the particle's bound states?

(b) For U large enough the Hamiltonian has two bound states. Sketch their wave functions, making it clear which describes the ground state.

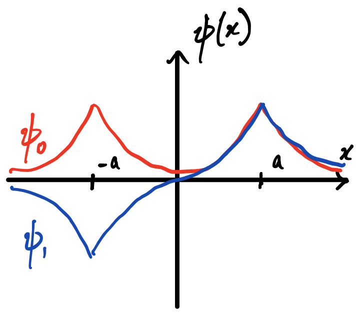
(c) For $U \leq U_c$ the Hamiltonian has only one bound state. Determine the value of U_c, in terms of the other parameters.

(a)



∵ ē Hamiltonian must be invariant under ē transformat^{n} $x \longrightarrow -x$, we must have energy eigenstates to obey this symmetry, up to a global phase. Thus, only purely odd or even funct^{n}'s are allowed for ē ground states.

(b)



Here, we have an even bound state ψ_0 & an odd bound state ψ_1. ψ_0 is ē ground state.

(c) We can use ē ansatz: $\psi(x) = \begin{cases} Ae^{kx} & , x < -a \\ Be^{kx} + Ce^{-kx} & , -a \leq x < b \\ De^{-kx} & , x > b \end{cases}$

ē boundary condit^{n}'s are: ① $Ae^{-ka} = Be^{-ka} + Ce^{ka}$ $\qquad \Delta \left.\frac{\partial \psi}{\partial x}\right|_a = -\frac{2mU}{\hbar^2}\psi(a)$

② $Be^{ka} + Ce^{-ka} = De^{-ka}$

③ $Ae^{-ka} - Be^{-ka} + Ce^{ka} = \frac{2mU}{\hbar^2} Ae^{-ka}$

④ $Be^{ka} - Ce^{-ka} + De^{-ka} = \frac{2mU}{\hbar^2} De^{-ka}$

① into ③: $2Ce^{ka} = \frac{2mU}{\hbar^2}(Be^{-ka} + Ce^{ka}) \Longrightarrow \left(2 - \frac{2mU}{\hbar^2}\right) Ce^{2ka} = \frac{2mU}{\hbar^2} B$

② into ④: $2Be^{ka} = \frac{2mU}{\hbar^2}(Be^{ka} + Ce^{-ka}) \Longrightarrow \frac{\hbar^2}{mU}\left(2 - \frac{2mU}{\hbar^2}\right) Ce^{2ka} = \left(4 - \frac{4mU}{\hbar^2}\right) Ce^{2ka} + \frac{2mU}{\hbar^2} Ce^{-2ka}$

For $\bar{c}$ antisymmetric bound state, we must have $B = -C$, so:

① $Ae^{-ka} = -2B\sinh(ka)$

② $2B\sinh(ka) = De^{-ka}$

③ $Ae^{-ka} - 2B\cosh(ka) = \frac{2mU}{\hbar^2} Ae^{-ka} \implies Ae^{-ka}\left(1 - \frac{2mU}{\hbar^2}\right) = 2B\cosh(ka)$

Into ①: $2B\cosh(ka) = -2B\left(1 - \frac{2mU}{\hbar^2}\right)\sinh(ka)$

$$\coth(ka) = \frac{2mU}{\hbar^2} - 1$$

$\coth(ka) = \frac{e^{ka} + e^{-ka}}{e^{ka} - e^{-ka}} \implies |\coth(ka)| > 1$

Thus, $\frac{2mU}{\hbar^2} > 2 \implies U_c = \frac{\hbar^2}{m}$, otherwise $\bar{c}$ antisymmetric solut^{n} does not exist.