4 Dec 2021

Problem 0, Page

1. A particle of mass m, with the Hamiltonian $H = \frac{p^2}{2m} + V(x)$, is moving in one dimension subject to an attractive potential of the form:

$$V(x) \;=\; -U\left[\delta(x + a/2) + \delta(x - a/2)\right]$$

with $U > 0$.

   (a) What consequences does the Hamiltonian's reflection symmetry have for the particle's bound states?

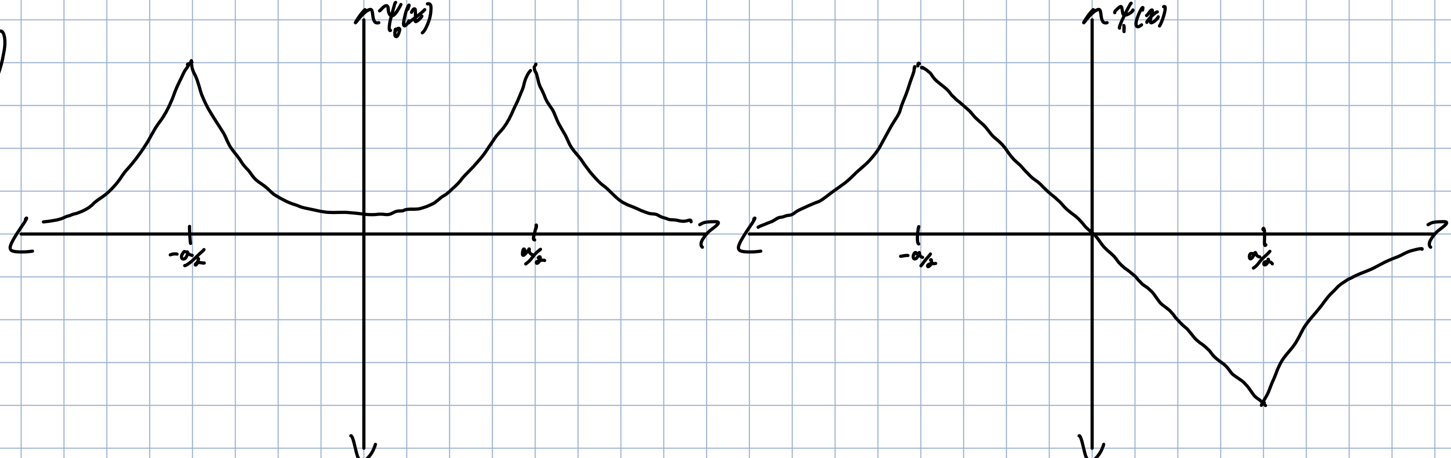
   (b) For U large enough the Hamiltonian has two bound states. Sketch their wave functions, making it clear which describes the ground state.

   (c) For $U \le U_c$ the Hamiltonian has only one bound state. Determine the value of U_c, in terms of the other parameters.

J14Q.1

a) It implies that the bound states are either symmetric or antisymmetric

b)



c) Once obtaining the W.F., an energy relation can be found from boundary conditions at the delta functions

define $l \equiv a/2$, $\kappa = \frac{\sqrt{-2mE}}{\hbar}$ ← $E<0$ for bound state

$$\psi(x) = \begin{cases} Ae^{\kappa x}, & x<-l \\ Be^{\kappa x} + Ce^{-\kappa x}, & |x|<l \\ De^{-\kappa x}, & x>l \end{cases}$$

Continuity at $x = \pm l$

$$Ae^{-\kappa l} = Be^{-\kappa l} + Ce^{\kappa l} \rightarrow A = B + Ce^{2\kappa l}$$

$$Be^{-\kappa l} = Be^{\kappa l} + Ce^{-\kappa l} \rightarrow D = Be^{2\kappa l} + C$$

$$\psi(x) = \begin{cases} Be^{\kappa x} + Ce^{\kappa(x+2l)}, & x<-l \\ Be^{\kappa x} + Ce^{-\kappa x}, & |x|<l \\ Be^{-\kappa(x-2l)} + Ce^{-\kappa x}, & x>l \end{cases} ; \quad \psi'(x) = \kappa \begin{cases} Be^{\kappa x} + Ce^{\kappa(x+2l)}, & x<-l \\ Be^{\kappa x} - Ce^{-\kappa x}, & |x|<l \\ -Be^{-\kappa(x-2l)} - Ce^{-\kappa x}, & x>l \end{cases}$$

$$-\frac{\hbar^2}{2m}\psi''(x) - U\delta(x+\ell)\psi(x) - U\delta(x-\ell)\psi(x) = E\psi(x)$$

$$\lim_{\varepsilon\to 0}\int_{\pm\ell-\varepsilon}^{\pm\ell+\varepsilon} -\frac{\hbar^2}{2m}\psi''(x) - U\delta(x+\ell)\psi(x) - U\delta(x-\ell)\psi(x)\,dx = \lim_{\varepsilon\to 0}\int_{\pm\ell-\varepsilon}^{\pm\ell+\varepsilon} E\psi(x)\,dx$$

$$-\frac{\hbar^2}{2m}\left[\psi'(\pm\ell^+) - \psi'(\pm\ell^-)\right] - U\psi(\pm\ell) = 0$$

$$\psi(\pm\ell) = -\frac{\hbar^2}{2mU}\left[\psi'(\pm\ell^+) - \psi'(\pm\ell^-)\right]$$

$$\psi'(\ell^+) - \psi'(\ell^-) = \kappa\left[(-Be^{\kappa\ell} - Ce^{-\kappa\ell}) - (Be^{\kappa\ell} - Ce^{-\kappa\ell})\right] = -2\kappa Be^{\kappa\ell}$$

$$\psi'(-\ell^+) - \psi'(-\ell^-) = \kappa\left[(Be^{-\kappa\ell} - Ce^{\kappa\ell}) - (Be^{-\kappa\ell} + Ce^{\kappa\ell})\right] = -2\kappa Ce^{\kappa\ell}$$

at $x = -\ell$:

$$Be^{-\kappa\ell} + Ce^{\kappa\ell} = -\frac{\hbar^2}{2mU}(-2\kappa Ce^{\kappa\ell}) = \frac{\hbar^2\kappa}{mU}Ce^{\kappa\ell}$$

$$B = C\left(\frac{\hbar^2\kappa}{mU} - 1\right)e^{2\kappa\ell}$$

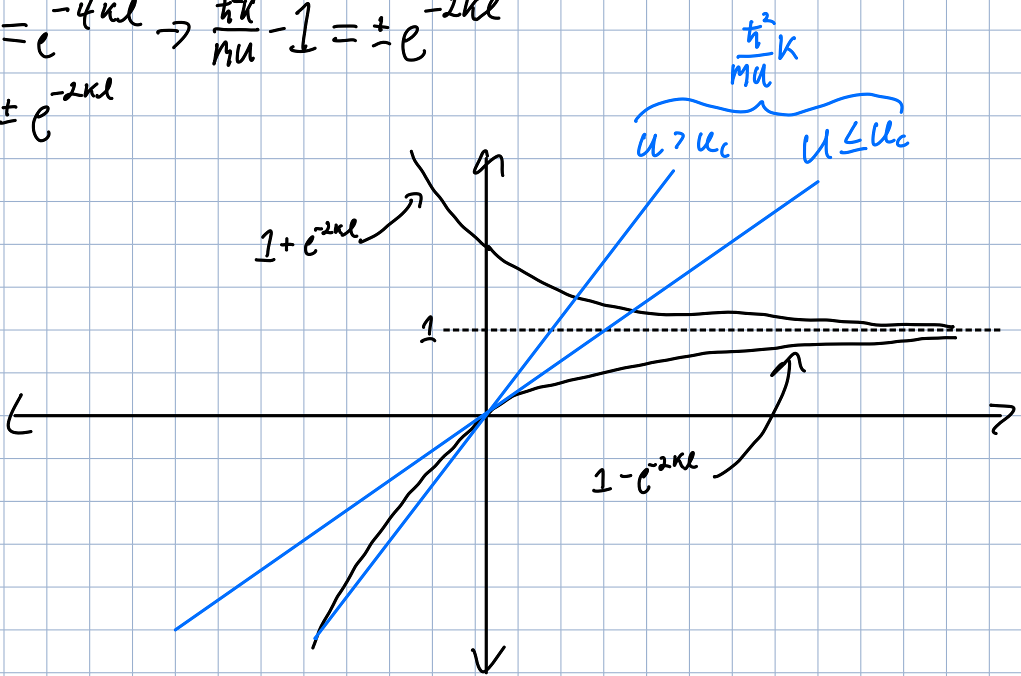
at $x = \ell$:

$$Be^{\kappa\ell} + Ce^{-\kappa\ell} = -\frac{\hbar^2}{2mU}(-2\kappa Be^{\kappa\ell}) = \frac{\hbar^2\kappa}{mU}Be^{\kappa\ell}$$

$$C = B\left(\frac{\hbar^2\kappa}{mU} - 1\right)e^{2\kappa\ell} = C\left(\frac{\hbar^2\kappa}{mU} - 1\right)^2 e^{4\kappa\ell}$$

$$\left(\frac{\hbar^2\kappa}{mU} - 1\right)^2 = e^{-4\kappa\ell} \rightarrow \frac{\hbar^2\kappa}{mU} - 1 = \pm e^{-2\kappa\ell}$$

$$\frac{\hbar^2}{mU}\kappa = 1 \pm e^{-2\kappa\ell}$$



Need to find when $\frac{\hbar^2}{mu}\kappa = 1 - e^{-2\kappa l}$ for $\kappa < 0$

take for high-energy state where $E \approx 0 \rightarrow \kappa << 1$

$$\frac{\hbar^2}{mu_c}\kappa = 1 - e^{-2\kappa l} = 1 - (1 - 2\kappa l) = 2\kappa l = \kappa a$$

$$U_c = \frac{\hbar^2}{ma}$$

$$\boxed{U_c = \frac{\hbar^2}{ma}}$$