J14Q.3 (Solution by Jim Wu)
Consider a quantum system consisting of a harmonic oscillator that is coupled to a spin-1/2 particle. The Hamiltonian is given by

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + \hbar\Omega S_z + \hbar g(aS_+ + a^\dagger S_-)$$

where $a, a^\dagger, S_z, S_+, S_-$ are the usual quantum operators for a harmonic oscillator and spin-1/2 particle.
When $g = 0$, the eigenstates of the Hamiltonian can be labeled by $|n, \pm\rangle$, where n is the harmonic oscillator occupation number and $+$ and $-$ refers to spin up and down states.

(a) Determine which of the uncoupled states of the Hamiltonian mix together when $g \neq 0$.

(b) Find the eigenstates of the Hamiltonian when $g \neq 0$ without making any assumptions about the relative size of various terms in (1).

(c) Make a sketch of how the energy levels change as a function of Ω in the range $0 < \Omega/\omega < 2$. Assume moderate coupling strength $g < \omega$.

Solution:

(a) Upon applying the Hamiltonian operator on the state $|n, +\rangle$, we find that the we get a mix between a $|n, +\rangle$ and a $|n+1, -\rangle$ state due to the $a^\dagger S_-$ operator. On the other hand, if we apply the Hamiltonian to the $|n, -\rangle$ state for $n \neq 0$, then we get a mix between a $|n, -\rangle$ and a $|n-1, +\rangle$ state due to the aS_+ operator. So, the eigenstates of the Hamiltonian are of the form

$$\begin{aligned} |\psi_1\rangle &= c_1 |n, +\rangle + c_2 |n+1, -\rangle \\ |\psi_2\rangle &= c_3 |n, -\rangle + c_4 |n-1, +\rangle \end{aligned}$$

and $|0, -\rangle$, which is uncoupled to other states.

(b) Let's start by applying H on $|\psi_1\rangle$:

$$\begin{aligned} H|\psi_1\rangle &= a\left[\left(\hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar\Omega}{2}\right)|n, +\rangle + \hbar g\sqrt{n+1}\,|n+1, -\rangle\right] \\ &\quad + b\left[\left(\hbar\omega\left(n + \frac{3}{2}\right) - \frac{\hbar\Omega}{2}\right)|n+1, -\rangle + \hbar g\sqrt{n+1}\,|n, +\rangle\right] \\ &= \left[\hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar\Omega}{2} + \frac{b}{a}\hbar g\sqrt{n+1}\right] a\,|n, +\rangle \\ &\quad + \left[\hbar\omega\left(n + \frac{3}{2}\right) - \frac{\hbar\Omega}{2} + \frac{a}{b}\hbar g\sqrt{n+1}\right] b\,|n+1, -\rangle \end{aligned}$$

For $|\psi_1\rangle$ to be an eigenstate, we must have $H|\psi\rangle = E|\psi\rangle$, requiring

$$\hbar\omega\left(n+\frac{1}{2}\right)+\frac{\hbar\Omega}{2}+\frac{b}{a}\hbar g\sqrt{n+1} = \hbar\omega\left(n+\frac{3}{2}\right)-\frac{\hbar\Omega}{2}+\frac{a}{b}\hbar g\sqrt{n+1}$$

$$\left(\frac{b^2-a^2}{ab}\right)\hbar g\sqrt{n+1} = \hbar(\omega-\Omega)$$

$$g\sqrt{n+1}a^2+(\omega-\Omega)ab-g\sqrt{n+1}b^2 = 0$$

Solving the quadratic equation for a, we find the ratio of the probability amplitude for $|n,+\rangle$ to $|n+1,-\rangle$ is

$$\frac{a}{b} = \frac{(\Omega-\omega)\pm\sqrt{(\omega-\Omega)^2+4g^2(n+1)}}{2g\sqrt{n+1}}$$

Substituting this back into $H|\psi\rangle$, we find that the energy eigenvalues are

$$\begin{aligned} E_1^{\pm} &= \hbar\omega\left(n+\frac{3}{2}\right)-\frac{\hbar\Omega}{2}+\frac{\hbar}{2}\left((\Omega-\omega)\pm\sqrt{(\omega-\Omega)^2+4g^2(n+1)}\right) \\ &= \hbar\omega\left(n+\frac{1}{2}\right)\pm\frac{\hbar\omega}{2}\sqrt{\left(1-\frac{\Omega}{\omega}\right)^2+4\left(\frac{g}{\omega}\right)^2(n+1)} \end{aligned}$$

If we do the same for $|\psi_2\rangle$, we get

$$\begin{aligned} H|\psi_2\rangle &= a\left[\left(\hbar\omega\left(n+\frac{1}{2}\right)-\frac{\hbar\Omega}{2}\right)|n,-\rangle+\hbar g\sqrt{n}\,|n-1,+\rangle\right] \\ &\quad + b\left[\left(\hbar\omega\left(n-\frac{1}{2}\right)+\frac{\hbar\Omega}{2}\right)|n-1,+\rangle+\hbar g\sqrt{n}\,|n,-\rangle\right] \\ &= \left[\hbar\omega\left(n+\frac{1}{2}\right)-\frac{\hbar\Omega}{2}+\frac{b}{a}\hbar g\sqrt{n}\right]a\,|n,+\rangle \\ &\quad + \left[\hbar\omega\left(n-\frac{1}{2}\right)+\frac{\hbar\Omega}{2}+\frac{a}{b}\hbar g\sqrt{n}\right]b\,|n-1,+\rangle \end{aligned}$$

and requiring that $|\psi_2\rangle$ be an eigenstate means that

$$\hbar\omega\left(n+\frac{1}{2}\right)-\frac{\hbar\Omega}{2}+\frac{b}{a}\hbar g\sqrt{n} = \hbar\omega\left(n-\frac{1}{2}\right)+\frac{\hbar\Omega}{2}+\frac{a}{b}\hbar g\sqrt{n}$$

$$\left(\frac{b^2-a^2}{ab}\right)\hbar g\sqrt{n} = \hbar(\Omega-\omega)$$

$$g\sqrt{n}a^2+(\Omega-\omega)ab-g\sqrt{n}b^2 = 0$$

Solving the quadratic equation yields the ratio of the probability amplitudes for $|n,-\rangle$ versus $|n-1,+\rangle$:

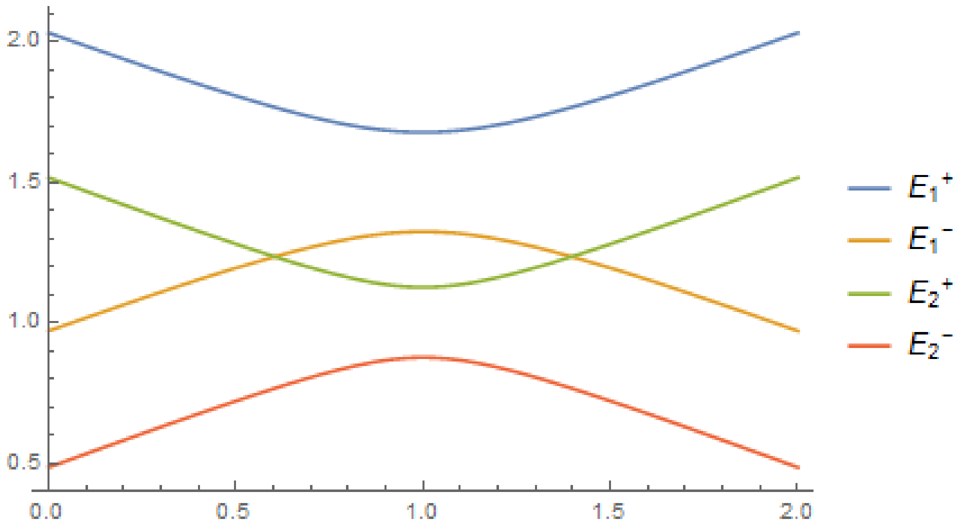
$$\frac{a}{b} = \frac{(\omega-\Omega)\pm\sqrt{(\Omega-\omega)^2+4g^2n}}{2g\sqrt{n}}$$

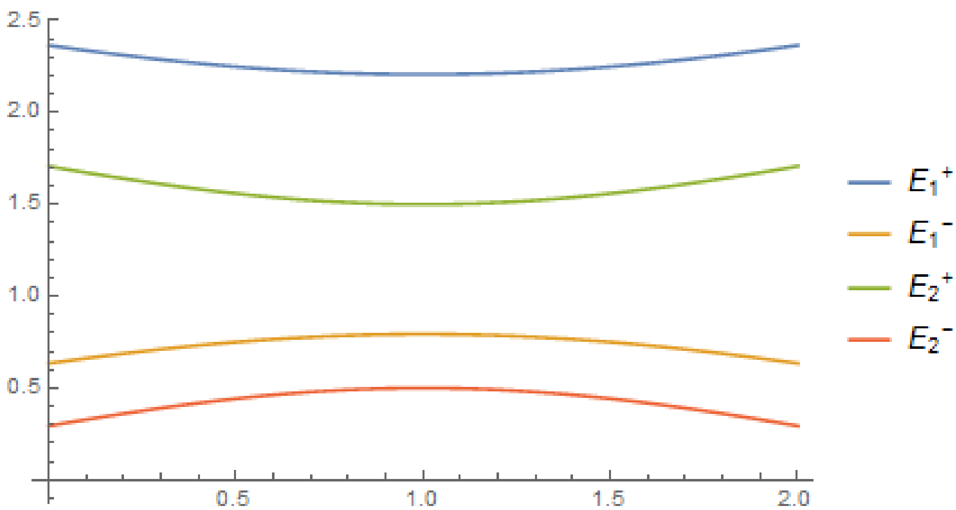
Substituting this back into the energy,

$$\begin{aligned} E_2^{\pm} &= \hbar\omega\left(n-\frac{1}{2}\right)+\frac{\hbar\Omega}{2}+\frac{\hbar}{2}\left((\omega-\Omega)\pm\sqrt{(\Omega-\omega)^2+4g^2n}\right) \\ &= \hbar\omega n\pm\frac{\hbar\omega}{2}\sqrt{\left(1-\frac{\Omega}{\omega}\right)^2+4\left(\frac{g}{\omega}\right)^2 n} \end{aligned}$$

(c) Clearly, $E_1^+ > E_1^-$ and $E_2^+ > E_1^-$ and in the range $0 < \Omega/\omega < 2$, the energy levels with $+$ superscripts are convex and the energy levels with $-$ subscripts are concave functions of Ω/ω. Furthermore, E_1^+ is clearly the largest and E_2^- is the smallest eigenvalue.

However, the relative size of E_1^- and E_2^+ depend on the both ratio Ω/ω and g/ω and sometimes the curves cross. Here are some example energy levels plotted in Mathematica as a function of Ω/ω at two different values of $g/\omega < 1$. For simplicity in graphing, I set $\hbar = 1$, $\omega = 1$, and $n = 1$.





On the top , we have $g/\omega = 1/8$ and on the bottom we have $g/\omega = 1/2$.

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