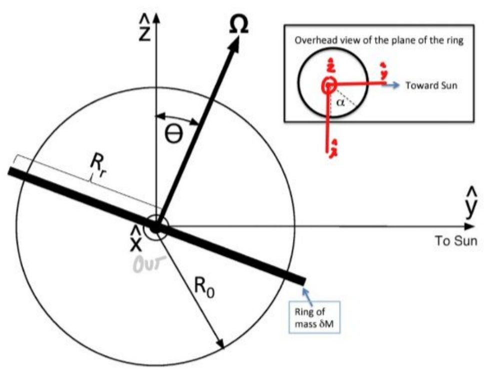
2. This is the dawning of the age of Aquarius, due to the precession of the Earth's spin axis $\vec{\Omega}$ around the celestial orbital axis $\hat{z}$. The Earth is slightly elliptical due to its spin. Approximate the Earth as a perfect sphere of radius R_0 and mass M, but assume a thin ring of radius R_r with mass δM is in the plane of the equator of the Earth. The Sun has mass M_s and is a distance R_{es} from the center of the Earth to the Sun. Assume that $R_r \ll R_{es}$, $\frac{R_r - R_0}{R_0} \ll 1$, and $\delta M \ll M$.



(a) From the figure, what is the torque $\vec{\tau}$ acting on the Earth about its center of mass due to the Sun? You'll need to do some approximations to get a tractable answer. One approximation is to use only the y coordinate in estimating how far each point on the ring is from the Sun.

(b) Neglecting the effects of the Moon's gravity, what is the rate of precession ω_p of the angular momentum $\mathbf{L}$ of the Earth around the celestial axis? If you couldn't solve part (a), then assume the torque is $\vec{\tau}$, where you don't need to know the magnitude of the vector $|\vec{\tau}| = \tau_p$, but you should know the direction. You may take the magnitude of the angular momentum, L, as known.

a) $\tau = \vec{r} \times \vec{F}$

$F_{gravity}$ $\quad \frac{GMm}{R_{es}^2}\hat{y}$ $\quad$ mass of ring

Each piece of mass $dm = \frac{\delta M\, d\alpha}{2\pi}$

Only ring experiences torque b/c symmetry of sphere on sphere

$\alpha = 0 \quad \hat{r} = \hat{x}$

$\alpha = \frac{\pi}{2}, \quad \hat{r} = \cos\theta\, \hat{y} - \hat{z}\sin\theta$

$\alpha = \pi \quad \hat{r} = -\hat{x}$

$\alpha = \frac{3\pi}{4} \quad \hat{r} = -\cos\theta\, \hat{y} + \hat{z}\sin\theta$

$\hat{r} = \hat{x}(\cos\alpha) + \hat{y}(\cos\theta \sin\alpha) + \hat{z}(-\sin\theta\sin\alpha)$

$\vec{r}_{dist} = R_0 \hat{r}$ $\quad$ Say the distance to the Sun only considers the y coordinate

$$dF_g = \frac{GM\,dm}{(R_{es} - R_0 \sin\alpha\cos\theta)^2}\hat{y} \approx \frac{GM\,dm}{R_{es}^2}\left(1 + 2\frac{R_0}{R_{es}}\sin\alpha\cos\theta\right)$$

$$\tau = \int \vec{r}\times d\vec{F}_g = \int_0^{2\pi} R_0 \Big(\underbrace{(\cos\alpha, \cos\theta\sin\alpha, -\sin\theta\sin\alpha)\times\hat{y}}_{\hat{x}\sin\theta\sin\alpha + \hat{z}(\cos\alpha)}\Big)\frac{GM\delta m}{2\pi R_{es}^2}\,d\alpha\left(1 + 2\frac{R_0}{R_{es}}\sin\alpha\cos\theta\right)$$

$$\begin{vmatrix} x & y & z \\ \hat{x} & \hat{y} & \hat{z} \\ 0 & 1 & 0 \end{vmatrix}$$

$$\tau_z = R_0\int_0^{2\pi}\cos\alpha\, d\alpha\left(\frac{GM\delta m}{2\pi R_{es}^2}\right)\left(1 + \frac{2R_0}{R_{es}}\sin\alpha\cos\theta\right) = 0$$

Integrates to 0 $\quad$ & $\int_0^{2\pi}\sin\alpha\cos\alpha\, d\alpha = \int_0^0 u\,du = 0$

$$\tau_x = R_0\int_0^{2\pi}\sin\theta\sin\alpha\left(\frac{GM\delta m}{2\pi R_{es}^2}\right)\left(1 + \frac{2R_0}{R_{es}}\sin\alpha\cos\theta\right)$$

$$= R_0\left(\frac{GM\delta m}{\cancel{2\pi} R_{es}^2}\right)\left(\frac{\cancel{2}R_0}{R_{es}}\right)\sin\theta\cos\theta\underbrace{\int_0^{2\pi}\sin^2\alpha\, d\alpha}_{\cancel{\pi}}$$

$$\tau_x = \frac{GM\delta m}{R_{es}}\left(\frac{R_0^2}{R_{es}^2}\right)\sin\theta\cos\theta$$

b) $\vec{\tau} = \frac{dL}{dt}$ $\tau_x = \left(\frac{dL}{dt}\right)_x = \frac{dL_x}{d\phi}\frac{d\phi}{dt} = L\sin\theta\frac{d\phi}{dt} = L\omega_p\sin\theta$

Sphere Approximateley

$L = I\Omega = \frac{2}{5}MR^2\Omega$

total

$$\omega_p = \frac{\tau_x}{\frac{2}{5}MR_0^2\Omega\sin\theta}$$

$$\Omega_p = \frac{5}{2}\frac{G\delta M}{R_{es}}\left(\frac{R_0^2}{R_{es}^2}\right)\cos\theta\,\frac{1}{R_0^2\Omega} = \frac{5}{2}\frac{G\delta M}{R_{es}^3\Omega}\cos\theta$$