J14M.1 Solution

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a) We can write down the total energy right after the impulse and at the periapsis

$$\begin{aligned} E_i =& \frac{1}{2}mv^2 - \frac{GMm}{R} \\ E_f =& \frac{1}{2}mv_p^2 - \frac{GMm}{R/5} \\ \Rightarrow \frac{1}{2}mv^2 - \frac{GMm}{R} =& \frac{1}{2}mv_p^2 - \frac{GMm}{R/5} \\ \Rightarrow v_p^2 =& v^2 + \frac{8GM}{R} \end{aligned}$$

Unfortunately they don't give us the mass of the planet, M. Fortunately, that's one of the few things that stays the same between the circular orbit and the elliptical orbit. So let's come up with an equation for M using the old circular orbit. First, we write down the effective potential

$$V_{eff} = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

Where $L = mvr$. The minimum of this will give us a formula for R.

$$\begin{aligned} \frac{dV_{eff}}{dr} = 0 =& -\frac{L^2}{mr^3} + \frac{GMm}{r^2} \\ r_{circular} = R =& \frac{L^2}{GMm^2} \\ \Rightarrow M =& \frac{L^2}{GRm^2} \\ M =& \frac{(mvR)^2}{GRm^2} \\ M =& \frac{v^2R}{G} \end{aligned}$$

Plugging this expression for M back into the expression for v_p we get

$$\begin{aligned} v_p^2 &= v^2 + 8v^2 \\ v_p &= 3v \end{aligned}$$

b) at the periapsis $\vec{r}$ is exactly perpendicular to $\vec{v}$ thus, using part a), we can get an expression for the angular momentum after the impulse.

$$\begin{aligned} \vec{L} &= \vec{r} \times \vec{p} \\ \vec{L} &= \frac{R}{5} \cdot 3mv\hat{z} \\ L &= \frac{3Rmv}{5} \end{aligned}$$

The angular momentum right after the impulse can be written, using the angle α, as

$$L = mvR\cos\alpha$$

Since this is still only a central force, the angular momentum is conserved so we get

$$\begin{aligned} \cos\alpha &= \frac{3}{5} \\ \alpha &= \cos^{-1}\left(\frac{3}{5}\right) \end{aligned}$$