J14M.1 Solution

Eric Emdee

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a) We can write down the total energy right after the impulse and at the periapsis

\[ E_i = \frac{1}{2}mv^2 - \frac{GMm}{R} \]

\[ E_f = \frac{1}{2}mv_p^2 - \frac{GMm}{R/5} \]

\[ \Rightarrow \frac{1}{2}mv^2 - \frac{GMm}{R} = \frac{1}{2}mv_p^2 - \frac{GMm}{R/5} \]

\[ \Rightarrow v_p^2 = v^2 + \frac{8GM}{R} \]

Unfortunately they don’t give us the mass of the planet, \( M \). Fortunately, that’s one of the few things that stays the same between the circular orbit and the elliptical orbit. So let’s come up with an equation for \( M \) using the old circular orbit. First, we write down the effective potential

\[ V_{eff} = \frac{L^2}{2mv^2} - \frac{GMm}{r} \]

Where \( L = mvr \). The minimum of this will give us a formula for \( R \).

\[ \frac{dV_{eff}}{dr} = 0 = -\frac{L^2}{mvr^3} + \frac{GMm}{r^2} \]

\[ r_{circular} = R = \frac{L^2}{GMm^2} \]

\[ \Rightarrow M = \frac{L^2}{GRm^2} \]

\[ M = \frac{(mvR)^2}{GRm^2} \]

\[ M = \frac{v^2R}{G} \]
Plugging this expression for $M$ back into the expression for $v_p$ we get

\[ v_p^2 = v^2 + 8v^2 \]
\[ v_p = 3v \]

b) at the periapsis $\vec{r}$ is exactly perpendicular to $\vec{v}$ thus, using part a), we can get an expression for the angular momentum after the impulse.

\[ \vec{L} = \vec{r} \times \vec{p} \]
\[ \vec{L} = \frac{R}{5} \cdot 3m\hat{z} \]
\[ L = \frac{3Rmv}{5} \]

The angular momentum right after the impulse can be written, using the angle $\alpha$, as

\[ L = mvR \cos \alpha \]

Since this is still only a central force, the angular momentum is conserved so we get

\[ \cos \alpha = \frac{3}{5} \]
\[ \alpha = \cos^{-1} \left( \frac{3}{5} \right) \]