J14E.3 Solution

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December 7, 2016

This problem sucks because as far as I can tell it just wants the full derivation of the complex wave number in a conductive medium. Which is pretty difficult unless you read it in a text book, and if you did read it it's then asking if you memorized the derivation. We start with Maxwell's equations and Ohm's law. Also to make things more clear, I'm using ϵ_r for the dielectric constant and ϵ for the permittivity, likewise for the permeability.

$$\vec{J}_f = \sigma \vec{E} \tag{1}$$

$$\nabla \cdot \vec{E} = \frac{\rho_f}{\epsilon} \tag{2}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\nabla \times \vec{B} = \mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t} \tag{4}$$

Taking the cross product of both sides of (3) we get

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) \tag{5}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu \vec{J} + \mu\epsilon \frac{\partial \vec{E}}{\partial t}\right) \tag{6}$$

For a homogenous linear conducting medium, the free charge within the conductor will quickly dissipate to zero, thus reducing (2) to $\nabla \cdot \vec{E} = 0$. Using this and (1), equation (6) reduces to

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} \tag{7}$$

Then, using the form of the electric field given to us we can linearize this equation. However, the plane waves for these solutions have a complex wave number $\tilde{k}$. thus get

$$-\tilde{k}^2 \vec{E} = -i\mu\sigma\omega \vec{E} - \mu\epsilon\omega^2 \vec{E} \tag{8}$$

$$\tilde{k}^2 = i\mu\sigma\omega + \mu\epsilon\omega^2 \tag{9}$$

Where

$$\tilde{k} = k + i\kappa \tag{10}$$

k is the wavenumber of the wave (which will turn out to be ω dependent) while κ gives us the decay constant of the wave within the medium which will be necessary for part b). Squaring $\tilde{k}$ and separating the real and imaginary parts we get

$$k^2 - \kappa^2 = \mu\epsilon\omega^2 \tag{11}$$

$$-2k\kappa = \mu\sigma\omega \tag{12}$$

$$\Rightarrow k^2 - \left(\frac{\mu\sigma\omega}{2k}\right)^2 = \mu\epsilon\omega^2 \tag{13}$$

$$k^4 - \left(\frac{\mu\sigma\omega}{2}\right)^2 = \mu\epsilon\omega^2 k^2 \tag{14}$$

Then using the quadratic equation we get

$$k^2 = \frac{\mu\epsilon\omega^2 \pm \sqrt{(\mu\epsilon\omega^2)^2 + 4\left(\frac{\mu\sigma\omega}{2}\right)^2}}{2} \tag{15}$$

We know to take the plus solution since k was presupposed to be real. Thus the answer to part a) is

$$k(\omega) = \sqrt{\frac{\mu\epsilon\omega^2}{2}}\left(1 + \sqrt{1 + \frac{\sigma}{\epsilon\omega}^2}\right)^{1/2} \tag{16}$$

In terms of the given variables σ, ϵ_r and μ we get

$$k(\omega) = \sqrt{\frac{\mu\epsilon_r\epsilon_0\omega^2}{2}}\left(1 + \sqrt{1 + \frac{\sigma}{\epsilon_r\epsilon_0\omega}^2}\right)^{1/2} \tag{17}$$

b) For part b) we just continue the work from part a) by finding κ.

$$\left(\frac{\mu\sigma\omega}{2\kappa}\right)^2 - \kappa^2 = \mu\epsilon\omega^2$$

$$\left(\frac{\mu\sigma\omega}{2}\right)^2 - \kappa^4 = \mu\epsilon\omega^2\kappa^2$$

Once again using the quadratic equation and taking the solution which gives a real κ, we get

$$\kappa = \sqrt{\frac{\mu\epsilon\omega^2}{2}}\left(\sqrt{1 + \frac{\sigma}{\epsilon\omega}^2} - 1\right)^{1/2}$$

Thus the electric field is given by

$$\vec{E}_T(z) = E_{T0} e^{i(\tilde{k}z - \omega t)}$$
$$\vec{E}_T(z) = E_{T0} e^{-\kappa z} e^{i(kz - \omega t)}$$

Notice that $E_{T0} \neq E_i$. This is due to the change in wave amplitude from the transmission across the boundary. This is given by

$$E_{T0} = \frac{2}{1+\beta} E_i$$
$$\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \sqrt{\frac{\mu}{\mu_0 \epsilon_r}}$$

Which is a formula from Griffith's that's highly recommended to be memorized for prelims (along with the reflected wave amplitude equation). Thus the equation for the electric field amplitude is given by

$$E_T(z) = E_i e^{-\kappa z} \sqrt{\frac{\mu}{\mu_0 \epsilon_r}}$$

So at some distance d we get

$$E_T(d) = E_i e^{-\kappa d} \sqrt{\frac{\mu}{\mu_0 \epsilon_r}}$$

Where κ is (in terms of the given variables σ, ϵ_r and μ)

$$\kappa = \sqrt{\frac{\mu \epsilon_r \epsilon_0 \omega^2}{2}} \left(\sqrt{1 + \frac{\sigma}{\epsilon_r \epsilon_0 \omega}^2} - 1 \right)^{1/2}$$