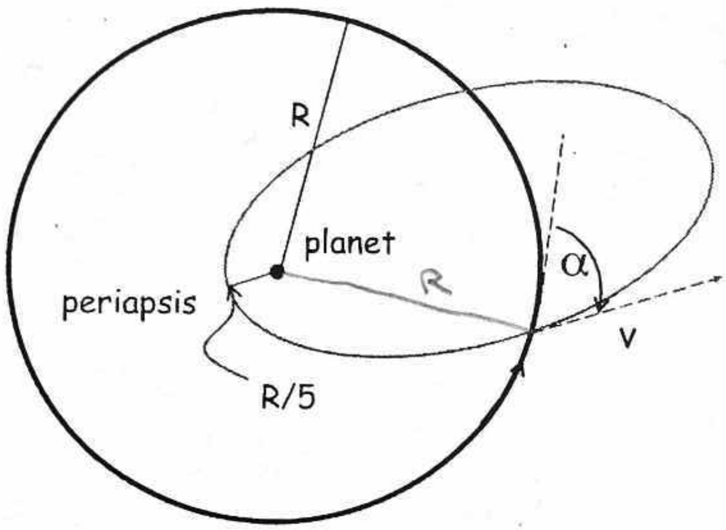
Section A. Mechanics

1. A satellite of mass m moves in a circular orbit of radius R about a much more massive planet (of unspecified mass). The satellite has speed v.

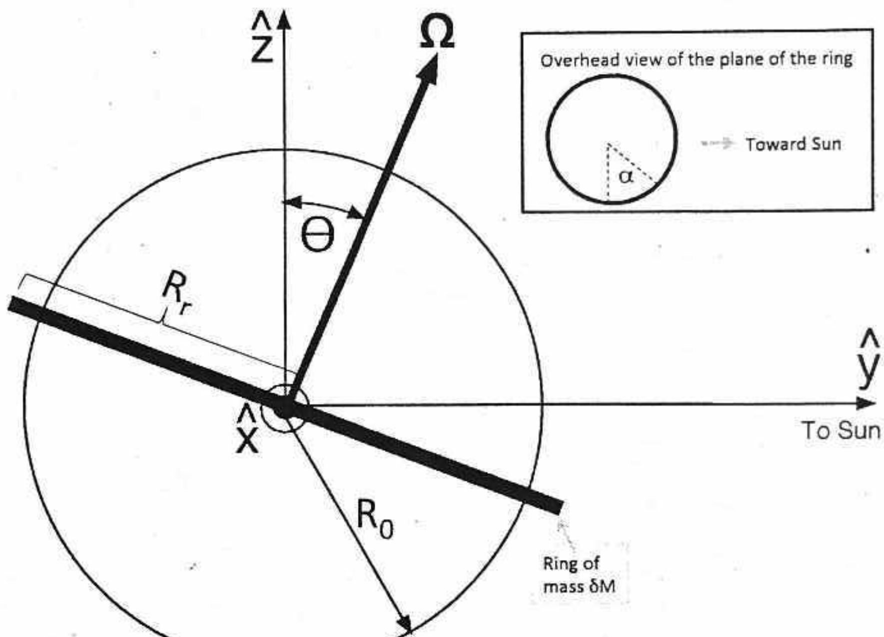


At a specific point in the satellite's circular orbit, the velocity of the satellite is abruptly rotated without changing the magnitude of its velocity. (The nature of this external impulse is not specified.) As shown in the figure, this causes the satellite to enter an elliptical orbit with its distance of closest approach = $R/5$. (This point in the elliptical orbit is called the periapsis in general.) The elliptical orbit is in the same plane as the circular orbit.

(a) What is the speed v_p of the satellite at the periapsis in terms of v?

(b) Through what angle α was the satellite turned? See figure for the definition of α.

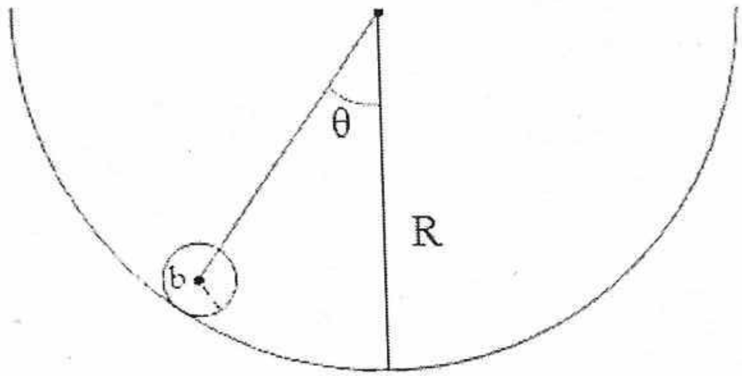
2. This is the dawning of the age of Aquarius, due to the precession of the Earth's spin axis $\vec{\Omega}$ around the celestial orbital axis $\hat{z}$. The Earth is slightly elliptical due to its spin. Approximate the Earth as a perfect sphere of radius R_0 and mass M, but assume a thin ring of radius R_r with mass δM is in the plane of the equator of the Earth. The Sun has mass M_s and is a distance R_{es} from the center of the Earth to the Sun. Assume that $R_r \ll R_{es}$, $\dfrac{R_r - R_0}{R_0} \ll 1$, and $\delta M \ll M$.



(a) From the figure, what is the torque $\vec{\tau}$ acting on the Earth about its center of mass due to the Sun? You'll need to do some approximations to get a tractable answer. One approximation is to use only the y coordinate in estimating how far each point on the ring is from the Sun.

(b) Neglecting the effects of the Moon's gravity, what is the rate of precession ω_p of the angular momentum $\mathbf{L}$ of the Earth around the celestial axis? If you couldn't solve part (a), then assume the torque is $\vec{\tau}$, where you don't need to know the magnitude of the vector $|\vec{\tau}| = \tau_p$, but you should know the direction. You may take the magnitude of the angular momentum, L, as known.

3. A marble of mass m and radius b rolls back and forth without slipping in a dish of radius R.



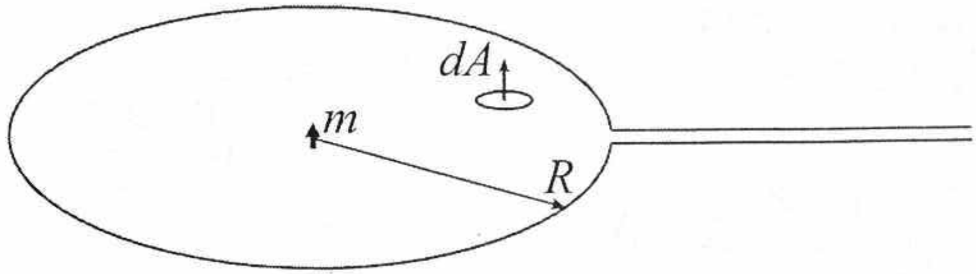
Find the frequency ω for small oscillations.

Section B. Electromagnetism

1. An infinite cylinder of radius R oriented parallel to the z-axis has uniform magnetization parallel to the x-axis, $\mathbf{M} = m_0\hat{x}$.

   Calculate the fields $\mathbf{H}$ and $\mathbf{B}$ everywhere inside and outside the cylinder. Sketch $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{M}$.

2. A particle with magnetic moment m is located at the center of a circular loop of wire with a radius R. The magnetic moment is pointing up, perpendicular to the plane of the loop.



(a) Calculate the flux of the magnetic field generated by the magnetic moment through the wire loop. Take the positive direction of an area element of the loop to point up.

The circular wire loop is now connected to a constant current source which maintains a constant current I flowing around the loop in the counter-clockwise direction when viewed from above.

(b) Calculate the mechanical work that needs to be done on the magnetic moment to rotate it by 180 degrees, turning it from pointing up to pointing down.

(c) Calculate the additional electrical energy supplied by the current source while the magnetic moment is being rotated. Comment on the conservation of energy.

3. A plane electromagnetic wave with $\vec{E} = E_i \exp(ikz - i\omega t)\hat{x}$ is incident from vacuum onto a weakly conductive medium with electrical conductivity σ, dielectric constant ϵ, and magnetic permeability μ. The medium extends through all space for $z > 0$.

   (a) Find the wavenumber $k(\omega)$ for the plane wave transmitted into the medium.

   (b) Find the electric field amplitude E_t of the wave transmitted into the medium at a distance $z = d$ inside the medium, accurate to first order in σ.

1. A particle of mass m, with the Hamiltonian $H = \frac{p^2}{2m} + V(x)$, is moving in one dimension subject to an attractive potential of the form:

$$V(x) = -U\left[\delta(x+a/2) + \delta(x-a/2)\right]$$

with $U > 0$.

   (a) What consequences does the Hamiltonian's reflection symmetry have for the particle's bound states?

   (b) For U large enough the Hamiltonian has two bound states. Sketch their wave functions, making it clear which describes the ground state.

   (c) For $U \leq U_c$ the Hamiltonian has only one bound state. Determine the value of U_c, in terms of the other parameters.

2. Scattering from a spherical potential:

   (a) Calculate the differential cross-section, $d\sigma/d\Omega$, for a particle of mass m scattering from a spherical potential $V(r) = V_0 e^{-(r/a)^2}$ using the first-order Born approximation. You may need the integral

   $$\int_0^\infty \sin r \, e^{-(r/b)^2} r dr = \frac{\sqrt{\pi}}{4} b^3 e^{-b^2/4}.$$

   (b) Calculate the total cross-section. It may be helpful to use the representation $|\vec{k} - \vec{k}'| = 2|\vec{k}| \sin(\theta/2)$, where θ is the angle between $\vec{k}$ and $\vec{k}'$.

   (c) What are the conditions on V_0, a and/or k for the first-order Born approximation to be valid?

3. Consider a quantum system consisting of a harmonic oscillator that is coupled to a spin-1/2 particle. The Hamiltonian is given by

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}) + \hbar\Omega S_z + \hbar g(aS_+ + a^\dagger S_-) \tag{1}$$

where $a, a^\dagger, S_z, S_+, S_-$ are the usual quantum operators for a harmonic oscillator and spin-1/2 particle.

When $g = 0$, the eigenstates of the Hamiltonian can be labeled by $|n, \pm\rangle$, where n is the harmonic oscillator occupation number and $+$ and $-$ refers to spin up and down states.

(a) Determine which of the uncoupled states of the Hamiltonian mix together when $g \neq 0$.

(b) Find the eigenstates of the Hamiltonian when $g \neq 0$ without making any assumptions about the relative size of the various terms in (1).

(c) Make a sketch of how the energy levels change as a function of Ω in the range $0 < \Omega/\omega < 2$. Assume moderate coupling strength $g < \omega$.

Section B. Statistical Mechanics and Thermodynamics

1. Consider a liquid placed in a very wide container that is in thermal equilibrium at temperature T with its surroundings. Let $z(\vec{r})$ be the height of the liquid at point $\vec{r} = (x, y)$ defined such that the equilibrium height in absence of thermal fluctuations is $z(\vec{r}) = 0$. For small deviations around the equilibrium, the potential energy is approximately

$$E_{\text{pot}} \approx E_0 + \frac{1}{2}\int dxdy\left[\sigma\left(\frac{\partial z}{\partial x}\right)^2 + \sigma\left(\frac{\partial z}{\partial y}\right)^2 + \rho g z^2\right],$$

where E_0 is a constant, σ is the surface tension, ρ is the difference between the density of the liquid and that of the gas, and g is the gravitational acceleration.

(a) For a periodic box of side length L, express the potential energy E_{pot} in terms of the Fourier coefficients $A(\vec{k})$ defined by

$$z(\vec{r}) = \frac{1}{L}\sum_{\vec{k}} e^{i\vec{k}\cdot\vec{r}} A(\vec{k}),$$

where $A(-\vec{k}) = A(\vec{k})^*$ and $\vec{k} = (k_x, k_y) = \frac{2\pi}{L}(n_x, n_y)$ (with n_x and n_y integers).

(b) Due to thermal fluctuations,

$$\left\langle |A(\vec{k})|^2 \right\rangle = \frac{1}{a\vec{k}^2 + b},$$

as long as $|\vec{k}|$ is below a certain cutoff. What are the values of a and b at temperature T, in terms of the model's parameters (σ, ρ, T, L)?

(c) Find an approximate expression for the r.m.s. width $W = \sqrt{\langle z(\vec{r})^2 \rangle}$, for wide containers, in terms of a, b, and the maximal value $k_{\max}$ of $|\vec{k}|$. Assume also that $k_{\max}^2 \gg b/a$.

Hint: modes with different wavevectors are not correlated, and thus $\left\langle A(\vec{k})A(\vec{k}')^* \right\rangle = 0$ if $\vec{k} \neq \vec{k}'$.

(d) What determines $k_{\max}$?

2. Consider a Fermi gas of N non-interacting particles in d dimensions where each particle has kinetic energy K.E. $= a|\vec{p}|^{\nu}$. The Fermi gas is placed in a box of volume V. Here, a and ν are positive constants, and N is assumed to be very large.

   (a) The Fermi energy can be written approximately as $E_F \approx \gamma N^{\lambda}$ for some γ and λ. Determine the exponent λ in terms of d and ν.

   (b) How does the heat capacity scale with temperature and the number of particles at small temperatures? Give the answer in terms of λ.

   (c) For this Fermi gas at temperature $T > 0$ the pressure P is related to the total energy E through $P = \alpha E/V$. Find α in terms of ν and d.

   Hint: P may be expressed through an appropriate derivative of the partition function. Think about how the energy of any given state changes with V.

   (d) For a relativistic Fermi gas in 3 dimensions $\nu = 1$. For this case derive $P = \alpha E/V$ also from the kinetic theory, with P expressed as the force per unit area exerted by the gas particles on the walls of the container.

3. Organic polymers are modeled as flexible chains whose links are rigid segments of length b that can pivot freely relative to each other. In the random walk approximation, the effects of overlaps between the links are ignored and the polymer configurations are taken to resemble random paths of N steps.

   Take it as given that, for a simple random walk:
   i) the end to end distance $R(N)$ scales as $R(N) \approx b\sqrt{N}$.
   ii) the probability of landing at $\vec{R}$ after N steps (starting at the origin) is $\approx e^{-R^2/(2Nb^2)}$ (up to irrelevant pre-factors).

   (a) At the level of the random walk approximation, what is the *entropy* of the idealized polymer of N units with total length R?

   (b) The polymer's self energy is modeled by a (repulsive) energy $\lambda > 0$ for any two units that come within a fixed distance from each other (and zero contribution otherwise). Assuming that the polymer's units are spread relatively uniformly over a volume of diameter R, obtain an approximate expression for the polymer's *free energy* showing the dependence on λ, b, N, R and T (powers or simple functions). (You do not need to specify the constant coefficients.)

   (c) Minimizing the free energy, derive a relation of the form: $\langle R \rangle \approx N^{\nu}$ for the order of magnitude estimate of the equilibrium end-to-end distance of the self-repelling polymer (at fixed b and $\lambda > 0$), in d dimensions. What value does the approximate expression for the free energy yield for the *exponent* ν?