Prelims Solutions

Problem J13M2

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1

The Lagrangian of the system is $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$ so the equation for r is $\ddot{r} = \frac{l^2}{mr^3} - \frac{\partial V}{\partial r}$. For circular motion $\ddot{r} = 0 \rightarrow \frac{l^2}{mr^3} = \frac{\partial V}{\partial r}$. Since $\frac{l^2}{mr^3}$ is strictly positive, $\frac{\partial V}{\partial r} > 0$ is required for circular orbits. Denoting allowed circular orbits to have radii $r = \rho$ and plugging in $V(\rho) = -a\rho^2 + b\rho^4$ gives $-2a\rho + 4b\rho^3 > 0 \rightarrow \rho > \sqrt{\frac{a}{2b}}$. Intuitively, these are the radii outside the minimum circle (where $\frac{\partial V}{\partial r} = 0$) of this "Mexican hat" potential.

2

Assume we have a constant circular orbit of radius $r = \rho$ which satisfies $\ddot{r} = \ddot{\rho} = 0 = \frac{l^2}{m\rho^3} + 2a\rho - 4b\rho^3$. Now perturb the orbit by a small amount $\delta(t)$, $r = \rho + \delta$. Substituting into the equation for r gives to first order in δ:

$$\ddot{\delta} = \frac{l^2}{m\rho^3}(1 - 3\frac{\delta}{\rho}) + 2a\rho(1 + \frac{\delta}{\rho}) - 4b\rho^3(1 + 3\frac{\delta}{\rho}) = (\frac{-3l^2}{m\rho^4} + 2a - 12b\rho^2)\delta$$

Using the $0 = \frac{l^2}{m\rho^3} + 2a\rho - 4b\rho^3$ we can solve for $\frac{-3l^2}{m\rho^4} = 6a - 12b\rho^2$ and substitute into our equation for δ.

$$\ddot{\delta} = (6a - 12b\rho^2 + 2a - 12b\rho^2)\delta = (8a - 24b\rho^2)\delta = -\omega^2\delta$$

The perturbation is oscillatory/stable if $8a - 24b\rho^2 < 0 \rightarrow \frac{a}{3b} < \rho^2$. But $\rho^2 > \frac{a}{2b}$ to even have a circular orbits, so all circular orbits are stable.

3

Frequency of small oscillations is $\omega = \sqrt{24b\rho^2 - 8a} > \sqrt{4a}$.