Thermodynamics of superconductor

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PROBLEM

A metal has two phases, N (normal) and S (superconducting). Assume that in the normal phase the magnetization M (per unit volume) due to an applied external magnetic field H is negligible, so the magnetic induction or flux density $B = \mu_0(H + M) = \mu_0 H$ in the normal meta phase.

The metal is cooled down to a temperature T in a large magnetic field H and then H is reduced to zero. At temperature $T < T_c$, it is observed that as H is reduced there is a critical field

$$H_c(T) = H_0 \left[1 - \left(\frac{T}{T_c}\right)^2\right],$$

where a first-order phase transition from the normal state to the superconducting state occurs. The magnetic flux is completely expelled from the metal, and $B = 0$ in its interior (Meissner effect) for $H < H_c(T)$ in the superconducting state.

Recall that the magnetic variant of the Gibbs free energy (per unit volume) is $G(T, H) = U - T - M'H$, where $M' = \mu_0 M$ in SI units. (Ignore any thermal expansion of the metal, and treat its volume as fixed).

(a) Find the difference of the entropy densities $\Delta S(T) = S_N(T) - S_S(T)$ between the normal and the superconducting phasese (with the assumption of negligible magnetization in the normal phase, $S_N(T)$ is independent of H).

(b) If the ststem is heated in the absence of a magnetic field (at $H = 0$) it undergoes a continuous (second-order) phase transition from superconductor to normal metal at $T = T_c$. What is the discontinuity in its specific heat per unit volume at this phase transition? (Make a sketch showing how the specific heat varies with temperature near the transition). Which phase has the larger specific heat?

(c) By how much is the ground state ($T = 0$) energy per unit volume of the superconductor lower than that of the normal metal, when $H = 0$?

SOLUTION TO PART A

The Gibbs free energy for normal state and the superconductor state is the same on the phase boundary:

$$G_N(T, H) = G_S(T, H)$$

Now we take the total differential of this equation, and we get

$$\frac{\partial G_N}{\partial T} dT = \frac{\partial G_S}{\partial T} dT + \frac{\partial G_S}{\partial H} dH$$

We also know that for Gibbs free energy, the entropy and the magnetization are given by $S = -\partial_T G$, and $\mu_0 M = -\partial_H G$. So it is easy to find the entropy difference on the phase boundary

$$\Delta S(T) = S_N(T) - S_S(T) = \mu_0 M_S \frac{dH}{dT}$$

We know that in the superconducting phase the magnetization is $M_S = -H$. Also on the phase boundary we have

$$\frac{dH_c}{dT} = -H_0 \frac{2T}{T_c^2}$$

Take all of these thing into consideration, we get the entropy difference at phase transition point

$$\Delta S(T) = \frac{2\mu_0 H_0^2 T}{T_c^2}\left(1 - \frac{T^2}{T_c^2}\right).$$

SOLUTION TO PART B

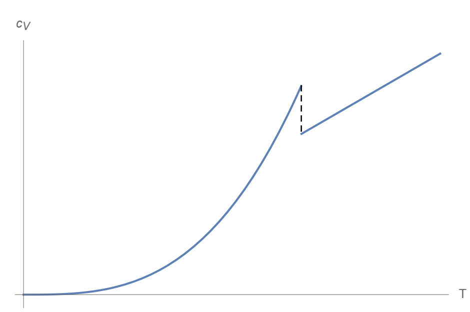
In the normal state, there is no magnetization at all, so the entropy doesn't have dependence on the external magnetic field H. For the superconducting state, by Maxwell relation, we have

$$\frac{\partial S_S}{\partial H} = -\frac{\partial^2 G_S}{\partial T \partial H} = \mu_0 \frac{\partial M}{\partial T}$$

But because of the Meissner effect, magnetization only depends on the external magnetic field $M = -H$, so $\partial_T M = 0$, which tells us that the entropy in superconducting state doesn't depend on external field either. So the entropy difference we get in part a is correct not only at the critical magnetic field. It is correct at any given magnetic field H. By the definition, we can find the heat capacity jump at the critical temperature without magnetic field:

$$\Delta c_V = c_V^N - c_V^S = T_c \frac{\partial \Delta S(T)}{\partial T}\bigg|_{T=T_c} = -\frac{4\mu_0 H_0^2}{T_c} < 0$$

That means at the phase transition point the heat capacity of superconducting state is higher than that of normal state. A brief sketch of the heat capacity is shown in the following figure:



SOLUTION TO PART C

When there is no external magnetic field, the Gibbs free energy is given by $G = U - TS$. Because the phase transition at zero field is first-order, the entropy of the two phases is the same, which tells us the internal energy U for the two phases are the same. Then the zero temperature difference at zero temperature is

$$\begin{aligned}
\Delta U &= \int_0^{T_c} \left[c_V^S(T) - c_V^N(T)\right] dT \\
&= -\int_0^{T_c} T \frac{\partial \Delta S(T)}{\partial T} dT \\
&= \int_0^{T_c} \Delta S(T) dT - T\Delta S(T)\Big|_{T=0}^{T=T_c} \\
&= \int_0^{T_c} \Delta S(T) dT \\
&= \frac{\mu_0 H_0^2}{2} \, .
\end{aligned}$$