(a) Total rotation of Earth can be written as:

\[ \mathbf{\dot{\omega}} = \omega_3 \hat{3} + \hat{\phi} (\cos \theta \hat{3} + \sin \theta \hat{1}) \]

(where \( \hat{1}, \hat{2}, \hat{3} \) are body frame axes)

\[ = (\omega_3 + \hat{\phi} \cos \theta) \hat{3} + \hat{\phi} \sin \theta \hat{1} \]

\( \Rightarrow \) Total translational KE of Earth is:

\[ T_{\text{rot}} = \frac{1}{2} I_3 (\omega_3 + \hat{\phi} \cos \theta)^2 + \frac{1}{2} I, \hat{\phi}^2 \sin^2 \theta \]

Denote mass of Earth by \( m \), \( \varepsilon \) is orbital angular velocity by \( \omega_0 \).

\[ L = \frac{1}{2} m r^2 + \frac{1}{2} m \omega_0^2 r^2 + \frac{GMm}{r} - \frac{GM (I_3 - I)}{2 \epsilon^3} \left( \frac{1}{2} - \frac{3}{2} \cos^2 \theta \right) \right. \]

\[ + \frac{1}{2} I_3 (\omega_3 + \hat{\phi} \cos \theta)^2 + \frac{1}{2} I, \hat{\phi}^2 \sin^2 \theta \]

\( \approx \) This is \( P_2 (\cos \theta) \) so it suggests \( \varepsilon \) torque is due to a quadrupole moment (i.e., oblateness of Earth).

(b) To obtain \( \hat{\phi} \), we should evaluate \( \varepsilon \) Euler-Lagrange equations.

Firstly, there is no otherwise stated information in \( \varepsilon \) question, I'm go\( ^2 \) to assume a circular orbit.

\( \Rightarrow \) \( i = 0 \), \( \omega_0 = \text{const} \), \( r = \text{const} \).

Now, for \( \varepsilon \) Euler-Lagrange equations:

\[ \phi \text{ eq}^2: \frac{d}{dt} \left[ I_3 (\omega_0 + \hat{\phi} \cos \theta)(\cos \theta) + I, \hat{\phi} \sin^2 \theta \right] = 0 \]

\[ \Rightarrow I_3 \cos \theta \dot{\omega}_3 + I_3 \omega_0 \dot{\phi} + I, \hat{\phi} \sin^2 \theta = 0 \]

But this is hard to work \( w/ \) so f**k it.

\[ \theta \text{ eq}^2: \text{Note } \# \text{ L is independent of } \hat{\phi} \text{ because we assumed no net } \hat{\phi} \text{.} \]

\[ \Rightarrow \frac{\partial L}{\partial \dot{\theta}} = 0 = \frac{3}{2} \frac{GM (I_3 - I)}{r^3} (-\cos \theta \sin \theta) + I_3 (\omega_0 + \hat{\phi} \cos \theta)(-\dot{\phi} \sin \theta) + I, \hat{\phi}^2 \sin^2 \theta \cos \theta \]

\[ \Rightarrow 0 = \frac{3GM (I_3 - I)}{2r^3} + \frac{I_3 \omega_0 \dot{\phi}}{\cos \theta} + I_3 \dot{\phi}^2 - I, \hat{\phi}^2 \]

Then, invoke fact \( \# \hat{\phi} \ll \omega_3 \) to drop \( \varepsilon \) last two terms.

\[ \Rightarrow \hat{\phi} \approx -\frac{3GM (I_3 - I) \cos \theta}{2I_3 \omega_3 r^3} \]
We also assumed this was a circular orbit, so: \( \omega_0^2 = \frac{GM}{r^3} \)

\[ \dot{\phi} \approx -\frac{3\omega_0^2 (I_1 - I_s) \cos \theta}{2I_s \omega_0} \]

\[ \frac{\dot{\phi}}{\omega_0} = -\frac{3\omega_0}{2I_s \omega_0} (I_1 - I_s) \cos \theta \]

(c) **Look** from above, we can see that \( \dot{\phi} \approx -\kappa \), where \( \kappa \) is a positive constant.

\[ \implies \dot{\phi} \text{ points downward, so the process is clockwise (given we are look from where an orbit looks CCW).} \]

This is opposite to orbital direction.

(d) As an estimate, \( \left| \frac{\dot{\phi}}{\omega_0} \right| \approx \frac{3}{2} \cdot 3 \times 10^{-3} \cdot \frac{T_s}{T_0} \cos(23.5^\circ) \), where \( \frac{T_s}{T_0} \) is just \( \frac{1}{365} \)

\[ \approx 4.5 \times 10^{-3} \cdot \frac{1}{365} \cos(23.5^\circ) \approx 10^{-5} \cdot 10^{-2} \cdot 1 \]

\[ \approx 10^{-5} \]

\[ \implies \frac{1 \text{ year}}{T_{\text{pre}}} \approx 10^{-5} \implies T_{\text{pre}} \approx 10^5 \text{ yr} > 26000 \text{ yr}. \]