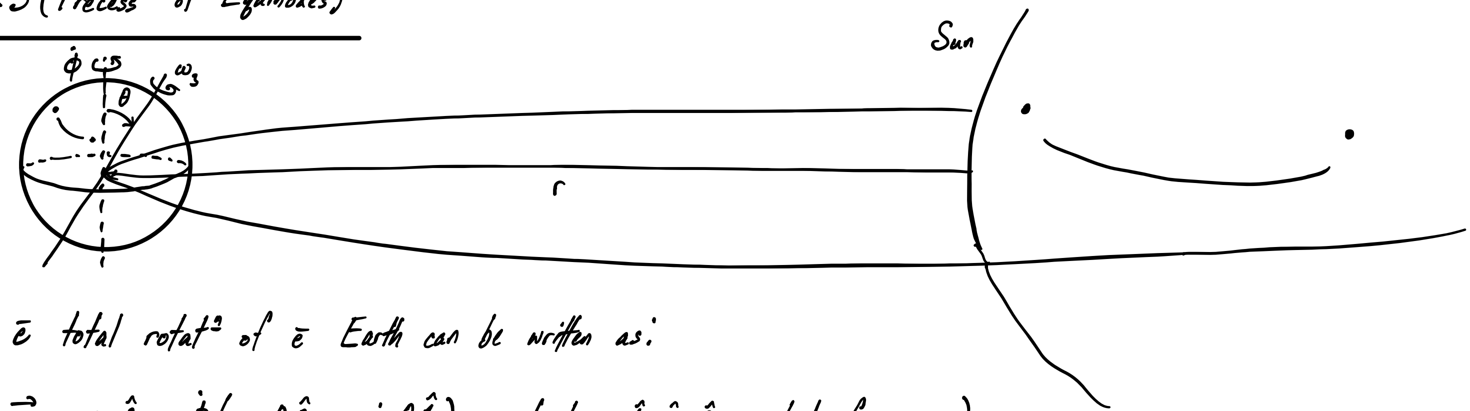
J13M.3 (Precess^n of Equinoxes)

(a)



ē total rotat^n of ē Earth can be written as:

$$\vec\omega = \omega_3\,\hat 3 + \dot\phi\left(\cos\theta\,\hat 3 + \sin\theta\,\hat 1\right) \qquad (\text{where } \hat 1, \hat 2, \hat 3 \text{ are body frame axes})$$

$$= \left(\omega_3 + \dot\phi\cos\theta\right)\hat 3 + \dot\phi\sin\theta\,\hat 1$$

$\Longrightarrow$ ē rotat^{n}al KE of ē Earth is:

$$T_{rot} = \frac{1}{2} I_3\left(\omega_3 + \dot\phi\cos\theta\right)^2 + \frac{1}{2} I_1 \dot\phi^2 \sin^2\theta$$

Denote ē mass of ē Earth by m, & ē orbital angular frequency by ω_0.

$$\mathcal{L} = \frac{1}{2} m\dot r^2 + \frac{1}{2} m r^2\omega_0^2 + \frac{GMm}{r} - \frac{GM(I_3 - I_1)}{2r^3}\left(\frac{1}{2} - \frac{3}{2}\cos^2\theta\right)$$

this is $P_2(\cos\theta)$ so it suggests ē torque is due to a quadrupole moment (i.e. oblateness of ē Earth).

$$+ \frac{1}{2} I_3\left(\omega_3 + \dot\phi\cos\theta\right)^2 + \frac{1}{2} I_1\dot\phi^2\sin^2\theta$$

(b) To obtain $\dot\phi$, we should evaluate ē Euler-Lagrange equat^{n}s.

Firstly, $\because$ there is no otherwise stated informat^n in ē quest^n, I'm go^g to assume a circular orbit.

$\Longrightarrow \dot r = 0$, $\omega_0 = \text{const}$, $r = \text{const}$.

Now, for ē Euler-Lagrange equat^{n}s:

ϕ eq^n: $\frac{d}{dt}\left[I_3\left(\omega_3 + \dot\phi\cos\theta\right)(\cos\theta) + I_1\dot\phi\sin^2\theta\right] = 0$

$$\Longrightarrow I_3\cos\theta\,\dot\omega_3 + I_3\cos^2\theta\,\ddot\phi + I_1\ddot\phi\sin^2\theta = 0$$

But this is hard to work w/ so fuck it.

θ eq^n: Note $\#$ $\mathcal{L}$ is independent of $\dot\theta$ because we assumed no nutat^n.

$$\Longrightarrow \frac{\partial\mathcal{L}}{\partial\theta} = 0 = \frac{3}{2}\frac{GM(I_3 - I_1)}{r^3}\left(-\cos\theta\sin\theta\right) + I_3\left(\omega_3 + \dot\phi\cos\theta\right)\left(-\dot\phi\sin\theta\right) + I_1\dot\phi^2\sin\theta\cos\theta$$

$$\Longrightarrow 0 = \frac{3GM(I_3 - I_1)}{2r^3} + \frac{I_3\omega_3\dot\phi}{\cos\theta} + I_3\dot\phi^2 - I_1\dot\phi^2$$

Then, invoke ē fact $\#$ $\dot\phi \ll \omega_3$ to drop ē last two terms.

$$\Longrightarrow \dot\phi \approx -\frac{3GM(I_3 - I_1)\cos\theta}{2I_3\omega_3 r^3}$$

We also assumed this was a circular orbit, so: $\omega_0^2 = \frac{GM_m}{r^3}$

$$\Longrightarrow \dot{\phi} \approx -\frac{3\omega_0^2 (I_3 - I_1)\cos\theta}{2I_3\,\omega_3}$$

$$\Longrightarrow \frac{\dot{\phi}}{\omega_0} \approx -\frac{3\omega_0}{2I_3\omega_3}(I_3 - I_1)\cos\theta$$

(c) Look^{g} from above, we can see $\#$ $\dot{\phi} \sim -\alpha$, where α is a posit^{v} const.

$\Longrightarrow \dot{\phi}$ points downward, so $\bar{e}$ precess^{n} is clockwise (given we are look^{g} from where $\bar{e}$ orbit looks CCW).

This is opposite $\bar{e}$ orbital direct^{n}.

(d) As an estimate, $\left|\frac{\dot{\phi}}{\omega_0}\right| \sim \frac{3}{2}\cdot 3\times 10^{-3}\cdot \frac{T_3}{T_0}\cos(23.5^\circ)$, where $\frac{T_3}{T_0}$ is just $\frac{1}{365}$

$$\sim 4.5\times 10^{-3}\cdot \frac{1}{365}\cos(23.5^\circ) \sim 10^{-3}\cdot 10^{-2}\cdot 1$$

$$\sim 10^{-5}$$

$$\Longrightarrow \frac{1\text{ year}}{T_{prec.}} \sim 10^{-5} \Longrightarrow T_{prec.} \sim 10^5\text{ yr} > 26000\text{ yr}$$ //.