We consider a projectile launched straight up with speed $v_0$ at some colatitude $\theta$. The Earth rotates toward the East with angular frequency $\omega$. Let us define orthogonal coordinates $\hat{z}$ - Up, $\hat{y}$ - North, $\hat{x}$ - East. Gravity is assumed to be constant and there is a friction force proportional to velocity.

It may be useful to recall the Coriolis force in coordinate free notation

$$F_c = -2m \omega \times v$$

We wish to find (a) the vertical velocity $v_z(t)$ as a function of time, ignoring Coriolis (b) $v_x(t), v_y(t), v_z(t)$ including Coriolis, but only two first order in $\omega$, and (c) the distance from launch to landing position. For simplicity we can assume terminal velocity $v_\infty$ is reached on the way down.

(a) Ignoring the Coriolis effect, the projectile would travel straight up and down. Newton’s law

$$m \ddot{z} = \sum F_z = -mg - b \dot{z}$$

can be rearranged

$$\dot{v}_z = -\frac{b}{m} \left( v_z + \frac{mg}{b} \right)$$

to find

$$v_z(t) = -\frac{mg}{b} + \left( v_0 + \frac{mg}{b} \right) e^{-bt/m}$$

Defining $v_\infty = mg/b$ to be the terminal velocity and $\alpha = b/m$ an inverse time constant, we see

$$v_z(t) = -v_\infty \left( 1 - e^{-\alpha t} \right) + v_0 e^{-\alpha t}$$

This expression agrees with intuition that $v_z(t)$ starts at $v_0$ and ends at $-v_\infty$. We see that the phrase ‘terminal velocity is reached’ can be interpreted as $e^{-\alpha t} \to 0$.

(b) To add the Coriolis force we should project the rotation vector onto coordinates

$$\vec{\omega} = \omega \begin{pmatrix} 0 \\ \sin \theta \\ \cos \theta \end{pmatrix}$$

and compute

$$\omega \times v = \omega \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \sin \theta & \cos \theta \\ v_x & v_y & v_z \end{pmatrix} = \omega \begin{pmatrix} v_z \sin \theta - v_y \cos \theta \\ v_x \cos \theta \\ -v_x \sin \theta \end{pmatrix} \approx \begin{pmatrix} v_z \omega \sin \theta \\ 0 \\ 0 \end{pmatrix}$$
Since $v_x, v_y$ both include factors of $\omega$, those terms become second order. To first order only $v_z$ remains. Now we see a system of differential equations

$$
\begin{align*}
    m\ddot{z} &= -b\dot{z} - mg \\
    m\ddot{y} &= -b\dot{y} \\
    m\ddot{x} &= -b\dot{x} - 2m\omega \sin \theta \dot{z}
\end{align*}
$$

The first equation is identical to that in part (a). The second equation is 0, because of initial condition $v_y(0) = 0$. The third equation is couples $\dot{v}_x$ to $v_z$ in an interesting way

$$
\dot{v}_x + \frac{b}{m} v_x = -2\omega \sin \theta v_z(t)
$$

This differential equation can be solved by Laplace Transform. ¹ The first step is to explicitly compute the Laplace transform of $v_z(t) = -v_\infty \left[ 1 - e^{-\alpha t} \left( 1 + \frac{v_0}{v_\infty} \right) \right]$

Setting $\eta = 1 + v_0/v_\infty$ we see

$$
\begin{align*}
    \tilde{v}_z(\gamma) &= \int_0^\infty v_z(t) e^{-\gamma t} dt \\
    &= -v_\infty \left( \int_0^\infty e^{-\gamma t} dt - \eta \int_0^\infty e^{-(\alpha+\gamma)t} dt \right) \\
    &= -v_\infty \left( \frac{1}{\gamma} - \frac{\eta}{\alpha + \gamma} \right)
\end{align*}
$$

Then the inverse transform yields

$$
v_x(t) = 2v_\infty \omega \sin \theta \left[ \frac{1}{2\pi i} \oint \frac{e^{\gamma t} d\gamma}{\gamma + \alpha} - \frac{\eta}{2\pi i} \oint \frac{e^{\gamma t} d\gamma}{(\gamma + \alpha)^2} \right]
$$

which can be solved as complex residues ²

$$
v_x(t) = 2v_\infty \omega \sin \theta \left[ \frac{1}{\alpha} - \frac{e^{-\alpha t}}{\alpha} - \eta te^{-\alpha t} \right] = 2v_\infty \omega \sin \theta \left[ 1 - e^{-\alpha t} (1 + \eta \alpha t) \right]
$$

One can readily check that this solution satisfies $v_x(0) = 0$ in addition to $\dot{v}_x + \alpha v_x = f(t)$.

¹Equations of the form $\dot{x} + \alpha x = f(t)$ can be Laplace transformed

$$
\tilde{x}(\gamma) = \int_0^\infty x(t) e^{-\gamma t} dt
$$

from here

$$
\tilde{x} = \frac{\tilde{f}}{\gamma + \alpha}
$$

and the solution is found by Inverse Laplace Transform

$$
x(t) = \frac{1}{2\pi i} \oint \tilde{x}(\gamma) e^{\gamma t} d\gamma
$$

Let us note that this calculation used initial value $v_x(0) = 0$ to obtain $\tilde{v}_x$.

²Some useful residue identities

$$
\oint \frac{dz}{z} = 2\pi i \quad \frac{1}{z-a} = \frac{1}{2\pi i} \oint \frac{f(z)dz}{z-a} = f(a) \quad \frac{1}{(z-a)^2} = f'(a)
$$
All together the solutions are to first order in $\omega$

\[v_x(t) = 2v_\infty \frac{\omega \sin \theta}{\alpha} \left[1 - e^{-\alpha t} (1 + \eta \alpha t)\right]\]

\[v_y(t) = 0\]

\[v_z(t) = -v_\infty (1 - \eta e^{-\alpha t})\]

with $\alpha = b/m$ and $\eta = 1 + v_0/v_\infty$. \(^3\)

(c) To find the $x$ displacement, we compute first the elapsed time $T$. This is constrained by $z$ displacement

$$\Delta z = \int_0^T v_z(t) \, dt = -v_\infty \left[ T - \frac{\eta}{\alpha} (1 - e^{-\alpha T}) \right] = 0$$

therefore we have total flight time

$$\alpha T = \eta (1 - e^{-\alpha T})$$

Now

$$\Delta x = \int_0^T v_x(t) \, dt$$

$$= 2v_\infty \frac{\omega \sin \theta}{\alpha} \left[ T - \frac{1 - e^{-\alpha T}}{\alpha} + \eta \left( T e^{-\alpha T} - \frac{1 - e^{-\alpha T}}{\alpha} \right) \right]$$

$$= 2v_\infty \frac{\omega \sin \theta}{\alpha} \left[ T - \frac{T}{\eta} + \eta T e^{-\alpha T} - T \right]$$

$$= 2v_\infty \frac{\omega \sin \theta}{\alpha} T e^{-\alpha T} \left( \frac{\eta - e^{\alpha T}}{\eta} \right)$$

$$= 2v_\infty \frac{\omega \sin \theta}{\alpha} T e^{-\alpha T} \left( \frac{\eta - 1}{\eta} + \frac{\alpha T}{\eta^2} \right)$$

This quantity can be either positive or negative depending on

$$\eta = 1 + \frac{v_0}{v_\infty}$$

which makes sense since $v_x$ drifts West as the projectile rises and East as the projectile falls, $\eta$ determines which motion it spends more time in.

For our case, we are given $b = mg/v_0$. It follows that $v_\infty = v_0$, $\alpha = g/v_0$ and $\eta = 2$. Numerically this gives $\alpha T = 1.593$. Although the question $\Delta x > 0$ is resolved, $e^{-1.593} \neq 1$ seems to contradict terminal velocity assumption...

\(^3\)Notice that as $e^{-\alpha t} \to 1$ and $v_y$ attains a terminal velocity, $v_x$ also appears to attain a terminal velocity. Although we approximated finite $t \in (0, T)$ as $\int_0^\infty dt$ in the Laplace transform, the behavior for large $\alpha t$ reaches steady state! This is a first order result, is it accurate? Well $\omega/\alpha$ enters our solution as a dimensionless ratio. $\alpha$ is the time it takes to reach terminal velocity, $\omega$ is the rotation of the Earth. Any reasonable flight will last a few minutes definitely less than an hour, while the rotation takes 24 hours, so $\omega/\alpha$ is at least an order of magnitude, and $O(\omega/\alpha)^2$ effects are at least 1/100. Although second order terms would couple $v_x$ into $v_y$ and $v_z$, this justifies the first order approximation.