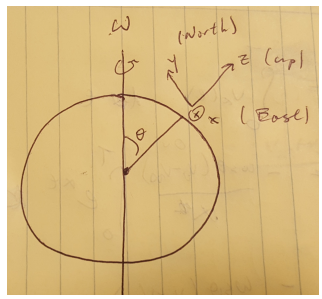
We consider a projectile launched straight up with speed v_0 at some co-latitude θ. The Earth rotates toward the East with angular frequency ω. Let us define orthogonal coordinates $\hat{z}$ - Up, $\hat{y}$ - North, $\hat{x}$ - East. Gravity is assumed to be constant and there is a friction force proportional to velocity.



It may be useful to recall the Coriolis force in coordinate free notation

$$F_c = -2m\,\omega \times v$$

We wish to find (a) the vertical velocity $v_z(t)$ as a function of time, ignoring Coriolis (b) $v_x(t), v_y(t), v_z(t)$ including Coriolis, but only two first order in ω, and (c) the distance from launch to landing position. For simplicity we can assume terminal velocity v_∞ is reached on the way down.

(a) Ignoring the Coriolis effect, the projectile would travel straight up and down. Newton's law

$$m\ddot{z} = \sum F_z = -mg - b\dot{z}$$

can be rearranged

$$\dot{v}_z = -\frac{b}{m}\left(v_z + \frac{mg}{b}\right)$$

to find

$$v_z(t) = -\frac{mg}{b} + \left(v_0 + \frac{mg}{b}\right)e^{-bt/m}$$

Defining $v_\infty = mg/b$ to be the terminal velocity and $\alpha = b/m$ an inverse time constant, we see

$$v_z(t) = -v_\infty\left(1 - e^{-\alpha t}\right) + v_0 e^{-\alpha t}$$

This expression agrees with intuition that $v_z(t)$ starts at v_0 and ends at $-v_\infty$. We see that the phrase 'terminal velocity is reached' can be interpreted as $e^{-\alpha t} \to 0$.

(b) To add the Coriolis force we should project the rotation vector onto coordinates

$$\vec{\omega} = \omega \begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix}$$

and compute

$$\omega \times v = \omega \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & \sin\theta & \cos\theta \\ v_x & v_y & v_z \end{pmatrix} = \omega \begin{pmatrix} v_z\sin\theta - v_y\cos\theta \\ v_x\cos\theta \\ -v_x\sin\theta \end{pmatrix} \approx \begin{pmatrix} v_z\omega\sin\theta \\ 0 \\ 0 \end{pmatrix}$$

Since v_x, v_y both include factors of ω, those terms become second order. To first order only v_z remains. Now we see a system of differential equation

$$\begin{aligned} m\ddot{z} &= -b\dot{z} - mg \\ m\ddot{y} &= -b\dot{y} \\ m\ddot{x} &= -b\dot{x} - 2m\omega \sin\theta \dot{z} \end{aligned}$$

The first equation is identical to that in part (a). The second equation is 0, because of initial condition $v_y(0) = 0$. The third equation is couples $\dot{v}_x$ to v_z in an interesting way

$$\dot{v}_x + \frac{b}{m} v_x = -2\omega \sin\theta v_z(t)$$

This differential equation can be solved by Laplace Transform. [1] The first step is to explicitly compute the Laplace transform of

$$v_z(t) = -v_\infty \left[1 - e^{-\alpha t} \left(1 + \frac{v_0}{v_\infty} \right) \right]$$

Setting $\eta = 1 + v_0/v_\infty$ we see

$$\begin{aligned} \tilde{v}_z(\gamma) &= \int_0^\infty v_z(t) e^{-\gamma t} dt \\ &= -v_\infty \left(\int_0^\infty e^{-\gamma t} dt - \eta \int_0^\infty e^{-(\alpha+\gamma)t} dt \right) \\ &= -v_\infty \left(\frac{1}{\gamma} - \frac{\eta}{\alpha + \gamma} \right) \end{aligned}$$

Then the inverse transform yields

$$v_x(t) = 2v_\infty \omega \sin\theta \left[\frac{1}{2\pi i} \oint \frac{e^{\gamma t} d\gamma}{\gamma(\gamma + \alpha)} - \frac{\eta}{2\pi i} \oint \frac{e^{\gamma t} d\gamma}{(\gamma + \alpha)^2} \right]$$

which can be solved as complex residues [2]

$$\begin{aligned} v_x(t) &= 2v_\infty \omega \sin\theta \left[\frac{1}{\alpha} - \frac{e^{-\alpha t}}{\alpha} - \eta t e^{-\alpha t} \right] \\ &= 2v_\infty \frac{\omega \sin\theta}{\alpha} \left[1 - e^{-\alpha t}(1 + \eta \alpha t) \right] \end{aligned}$$

One can readily check that this solution satisfies $v_x(0) = 0$ in addition to $\dot{v}_x + \alpha v_x = f(t)$.

[1] Equations of the form $\dot{x} + \alpha x = f(t)$ can be Laplace transformed

$$\tilde{x}(\gamma) = \int_0^\infty x(t) e^{-\gamma t} dt$$

from here

$$\tilde{x} = \frac{\tilde{f}}{\gamma + \alpha}$$

and the solution is found by Inverse Laplace Transform

$$x(t) = \frac{1}{2\pi i} \oint \tilde{x}(\gamma) e^{\gamma t} d\gamma$$

Let us note that this calculation used initial value $v_x(0) = 0$ to obtain $\tilde{v}_x$.

[2] Some useful residue identities

$$\oint \frac{dz}{z} = 2\pi i \qquad \frac{1}{2\pi i} \oint \frac{f(z)dz}{z - a} = f(a) \qquad \frac{1}{2\pi i} \oint \frac{f(z)dz}{(z-a)^2} = f'(a)$$

All together the solutions are to first order in ω

$$v_x(t) = 2v_\infty \tfrac{\omega \sin\theta}{\alpha} \left[1 - e^{-\alpha t}(1 + \eta\alpha t)\right]$$

$$v_y(t) = 0$$

$$v_z(t) = -v_\infty(1 - \eta e^{-\alpha t})$$

with $\alpha = b/m$ and $\eta = 1 + v_0/v_\infty$. [3]

(c) To find the x displacement, we compute first the elapsed time T. This is constrained by z displacement

$$\Delta z = \int_0^T v_z(t)dt = -v_\infty \left[T - \frac{\eta}{\alpha}(1 - e^{-\alpha T})\right] = 0$$

therefore we have total flight time

$$\alpha T = \eta(1 - e^{-\alpha T})$$

Now

$$\begin{aligned}
\Delta x &= \int_0^T v_x(t)dt \\
&= 2v_\infty \frac{\omega \sin\theta}{\alpha} \left[T - \frac{1 - e^{-\alpha T}}{\alpha} + \eta\left(Te^{-\alpha T} - \frac{1 - e^{-\alpha T}}{\alpha}\right)\right] \\
&= 2v_\infty \frac{\omega \sin\theta}{\alpha} \left[T - \frac{T}{\eta} + \eta T e^{-\alpha T} - T\right] \\
&= 2v_\infty \frac{\omega \sin\theta}{\alpha} T e^{-\alpha T} \left(\eta - \frac{e^{\alpha T}}{\eta}\right) \\
&= 2v_\infty \frac{\omega \sin\theta}{\alpha} T e^{-\alpha T} \left(\eta - \frac{1}{\eta} + \frac{\alpha T}{\eta^2}\right)
\end{aligned}$$

This quantity can be either positive or negative depending on

$$\eta = 1 + \frac{v_0}{v_\infty}$$

which makes sense since v_x drifts West as the projectile rises and East as the projectile falls, η determines which motion it spends more time in.

For our case, we are given $b = mg/v_0$. It follows that $v_\infty = v_0$, $\alpha = g/v_0$ and $\eta = 2$. Numerically this gives $\alpha T = 1.593$. Although the question $\Delta x > 0$ is resolved, $e^{-1.593} \neq 1$ seems to contradict terminal velocity assumption...

[3] Notice that as $e^{-\alpha t} \to 1$ and v_z attains a terminal velocity, v_x also appears to attain a terminal velocity. Although we approximated finite $t \in (0, T)$ as $\int_0^\infty dt$ in the Laplace transform, the behavior for large αt reaches steady state! This is a first order result, is it accurate? Well ω/α enters our solution as a dimensionless ratio. α is the time it takes to reach terminal velocity, ω is the rotation of the Earth. Any reasonable flight will last a few minutes definitely less than an hour, while the rotation takes 24 hours, so ω/α is at least an order of magnitude, and $O(\omega/\alpha)^2$ effects are at least 1/100. Although second order terms would couple v_x into $\dot{v}_y$ and $\dot{v}_z$, this justifies the first order approximation.