January 2013 Preliminary Exam, Electricity and Magnetism Problem 3

Kevin P. Nuckolls (k.nuckolls@princeton.edu)
Solution presented in class by Christian Jepsen

Problem: Complex Impedances

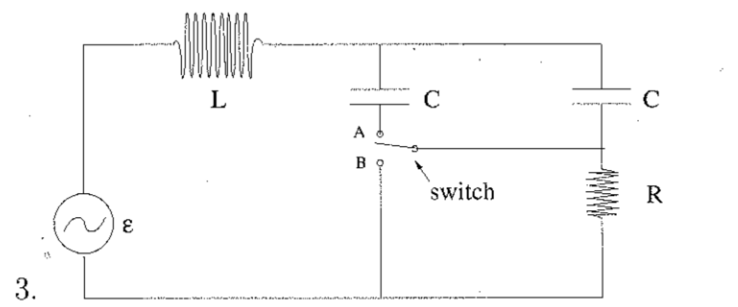

Figure 1: An AC circuit.

Figure 1: *Circuit diagram from J13E.3*

Consider the circuit above. The switch can be set in any of three positions, A, B, or open (unconnected). The source supplies a voltage $\epsilon(\omega) = \epsilon_0 e^{i\omega t}$.

(a) When the switch is connected to A, find the frequency ω that maximizes the current through the resistor R.

(b) If we then flip the switch to the B position, what is the average power dissipation in the circuit (ignoring transient effects).

(c) We now open the switch to the middle position. Find the value of the resistor R that will drop the amplitude of the current to $1/2$ the value you found in part (a), at the same frequency ω you found in part (a).

(d) Suppose that the inductor, of inductance L, is constructed from a solenoid with N turns over a length l.

Express the cross sectional area of the solenoid in terms of the inductance L, the number of turns N, the length l, and any fundamental constants.

Solution:

(a) When the switch is flipped to position A, the equivalent circuit is an LRC circuit, where the two capacitors form an effective capacitance in their parallel configuration of

$$C_{tot} = C + C = 2C \tag{1}$$

We want to consider this voltage, with its complex phase factor, with the system's complex impedances. The complex impedances of a resistor, capacitor, and inductors are $Z_R = R$, $Z_C = \frac{1}{i\omega C}$, and $Z_L = i\omega L$, respectively. Therefore,

$$\epsilon = I Z_{tot} \tag{2}$$

$$\implies |I|^2 = \frac{|\epsilon|^2}{|Z_{tot}|^2} = \frac{\epsilon_0^2}{R^2 + (\frac{1}{\omega C_{tot}} - \omega L)^2} = \frac{\epsilon_0^2}{R^2 + (\frac{1}{2\omega C} - \omega L)^2} \tag{3}$$

The current will be maximized for a frequency that makes the second factor in the denominator equal to zero.

$$\frac{1}{2\omega C} = \omega L \tag{4}$$

$$\omega = \sqrt{\frac{1}{2CL}} \tag{5}$$

(b) If the switch is now at position B, we short the resistor, which reroutes all of the current around it. Thus, $P = IV = 0$.

(c) Opening the switch also gives us an LRC circuit, but using only one of the capacitors. We want the value of R such that the current is half the answer from part (a), using the same frequency as part (a). Therefore,

$$|I_B|^2 = \frac{1}{4}|I_A|^2 = \frac{\epsilon_0^2}{R^2 + (\frac{1}{2\omega C} - \omega L)^2} = \frac{\epsilon_0^2}{4R^2} \tag{6}$$

$$\implies \left(\frac{1}{2\omega C} - \omega L\right)^2 = 3R^2 \tag{7}$$

$$\implies R = \sqrt{\frac{L}{6C}} \tag{8}$$

(d) The magnetic field in a solenoid of length l and N turns is $B = \mu_0 I \frac{N}{l}$. By Lenz's law,

$$|\epsilon| = \frac{d\Phi}{dt} = \frac{d}{dt}(B(NA)) = NA\frac{d}{dt}\left(\mu_0 I \frac{N}{l}\right) = \mu_0 \frac{N^2 A}{l}\dot{I} \equiv L\dot{I} \tag{9}$$

$$\implies L = \frac{\mu_0 N^2 A}{l} \implies A = \frac{Ll}{\mu_0 N^2} \tag{10}$$