J15E1 (Solution by Jim Wu)

(a) Let a current I circulate in a square wire of side d lying in the $x-y$ plane, with center at the origin. What is the vector potential $\vec{A}$ at a position x_0, where $x_0 \gg d$ (i.e. Taylor expand the denominator)?

(b) What is the magnetic field B at x_0?

(c) Now boost to a frame where the charge q_0 and the loop are both moving with speed $+v_0\hat{x}$. What is the electric field $\vec{E}'$ due to the loop acting on the charge (I need magnitude and direction). For this, you don't need the answer for part (b), just call the field B but you need direction.

(d) What is the total force $\vec{F}_{tot}$ acting on the charge q_0 in this frame?

Solution:

(a) Given a current distribution, the magnetic vector potential is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r'$$

where $\mathbf{r}'$ refers the position of points in the current source and $\mathbf{r}$ refers to the position of the observer.

For a square loop of current I, this vector potential reduces to

$$\mathbf{A} = \frac{\mu_0 I}{4\pi}\oint \frac{1}{\sqrt{r^2+r'^2-2\mathbf{r}\cdot\mathbf{r}'}}\, d\boldsymbol{\ell}'$$

where $\boldsymbol{\ell}'$ is the position vector of points in the square loop. Since the point of interest is at a position x_0 from the center of the loop, then, $r = x_0$ and we will compute everything from the center of the loop as the origin. From a Taylor expansion of the denominator, we get

$$\mathbf{A} = \frac{\mu_0 I}{4\pi x_0}\oint 1 + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{r} - \left(\frac{r'}{r}\right)^2 + \ldots\, d\boldsymbol{\ell}'$$

Taking on the first order approximation, we will neglect everything of higher order than r'/r. Note that the integral of $d\boldsymbol{\ell}'$ around the loop is zero as this is a closed contour integral of a vector. So, the approximate vector potential is

$$\mathbf{A} = \frac{\mu_0 I}{4\pi x_0^2}\oint \hat{\mathbf{r}}\cdot\mathbf{r}'\, d\boldsymbol{\ell}'$$

Note that the projection of $\mathbf{r}'$ onto $\hat{\mathbf{r}}$ is just the x' coordinate of the point on the loop. Assume. that the current goes in the counter-clockwise direction. Parameterizing $d\boldsymbol{\ell}$ as $dy'\hat{\mathbf{y}}$ for the vertical wires and $dx'\hat{\mathbf{x}}$ for the horizontal wires, and integrating around this loop in the same direction as the current, we get

$$\begin{aligned}
\mathbf{A} &= \frac{\mu_0 I}{4\pi x_0^2}\left[\int_{-d/2}^{d/2}\frac{d}{2}dy'\hat{\mathbf{y}} + \int_{d/2}^{-d/2}x'dx'\hat{\mathbf{x}} + \int_{d/2}^{-d/2}\left(-\frac{d}{2}\right)dy'\hat{\mathbf{y}}' + \int_{-d/2}^{d/2}x'dx'\hat{\mathbf{x}}\right] \\
&= \frac{\mu_0 I}{4\pi x_0^2}\left[\left(\frac{d}{2}\right)(d)\hat{\mathbf{y}} + 0 + \left(\frac{d}{2}\right)(d)\hat{\mathbf{y}} + 0\right] \\
&= \frac{\mu_0 I d^2}{4\pi x_0^2}\hat{\mathbf{y}}
\end{aligned}$$

Note that Id^2 is exactly the magnitude of the magnetic moment of the square wire. This magnetic vector potential is exactly the same as the dipole term from the multipole expansion, which is

$$\mathbf{A} = \frac{\mu_0(\mathbf{m}\times\hat{\mathbf{r}})}{4\pi r^2} = \frac{\mu_0}{4\pi x_0^2}(Id^2\hat{\mathbf{z}}\times\hat{\mathbf{x}}) = \frac{\mu_0 I d^2}{4\pi x_0^2}\hat{\mathbf{y}}$$

(b) The magnetic field can be obtained from taking the curl of the magnetic vector potential, $\mathbf{B} = \nabla\times\mathbf{A}$. Using the general vector potential of a magnetic dipole, we find that the magnetic field due to the square loop at $x_0 \gg d$ is approximately

$$\mathbf{B} = \frac{\mu_0(3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m})}{4\pi r^3} = \frac{\mu_0}{4\pi x_0^3}(3(Id^2\hat{\mathbf{z}}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}} - Id^2\hat{\mathbf{z}}) = -\frac{\mu_0 I d^2}{4\pi x_0^3}\hat{\mathbf{z}}$$

(c) Since the charge is at rest and there is no electric field, then the charge experiences no force.

(d) Let the S' frame be moving at velocity v in the x direction relative to the rest frame, S. Then the Lorentz transform from S to S' is

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v/c & 0 & 0 \\ -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Furthermore, the field tensor is

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

So, the electric field in the S' frame is

$$\begin{aligned}
E'_x/c = F'^{tx} &= \Lambda^t{}_\mu\Lambda^x{}_\nu F^{\mu\nu} = \Lambda^t{}_t\Lambda^x{}_x F^{tx} + \Lambda^t{}_x\Lambda^x{}_t F^{xt} = \gamma^2\left(1-\frac{v^2}{c^2}\right)(E_x/c) = E_x/c \\
E'_y/c = F'^{ty} &= \Lambda^t{}_\mu\Lambda^y{}_\nu F^{\mu\nu} = \Lambda^t{}_t\Lambda^y{}_y F^{tx} + \Lambda^t{}_x\Lambda^y{}_y F^{xy} = \gamma\left(\frac{E_y}{c} - \frac{v}{c}B_z\right) \\
E'_z/c = F'^{ty} &= \Lambda^t{}_\mu\Lambda^z{}_\nu F^{\mu\nu} = \Lambda^t{}_t\Lambda^z{}_z F^{tx} + \Lambda^t{}_x\Lambda^z{}_z F^{xz} = \gamma\left(\frac{E_z}{c} + \frac{v}{c}B_y\right)
\end{aligned}$$

So, in general, the electric field in the boosted frame is related to the fields in the fixed frame by

$$\vec{E}' = (E_x, \gamma(E_y - vB_z), \gamma(E_z + vB_y))$$

Since the charge and loop are moving with speed $+v_0\hat{x}$ in the boosted frame, then the frame S' must be moving $-v_0\hat{x}$ relative to the original fixed frame. Furthermore, given that there was no electric field in the S frame and that magnetic field from the loop acting on the charge was only in the $-\hat{z}$ direction, then

$$\vec{E}' = (0, -\gamma v_0 B_0, 0)$$

where $B_0 = \frac{\mu_0 I d^2}{4\pi x_0^3}$.

(e) We can also do the same analysis for $\vec{B}'$:

$$B'_x = F'^{yz} = \Lambda^y{}_\mu \Lambda^z{}_\nu F^{\mu\nu} = \Lambda^y{}_y \Lambda^z{}_z F^{yz} = B_x$$
$$B'_y = F'^{zx} = \Lambda^z{}_\mu \Lambda^x{}_\nu F^{\mu\nu} = \Lambda^z{}_z \Lambda^x{}_t F^{zt} + \Lambda^z{}_z \Lambda^x{}_x F^{zx} = \gamma\left(B_y + \frac{vE_z}{c^2}\right)$$
$$B'_z = F'^{xy} = \Lambda^x{}_\mu \Lambda^y{}_\nu F^{\mu\nu} = \Lambda^x{}_x \Lambda^y{}_y F^{xy} + \Lambda^x{}_t \Lambda^y{}_y F^{ty} = \gamma\left(B_z - \frac{vE_y}{c^2}\right)$$

For $v = -v_0$, $\vec{E} = 0$, and $\vec{B} = (0, 0, -B_0)$, the magnetic field in the S' frame is

$$\vec{B} = (0, 0, -\gamma B_0)$$

So, the net force on the charge in the S' frame is

$$\vec{F}' = q(\vec{E}' + \vec{v} \times \vec{B}') = q(-\gamma v_0 B_z \hat{y} + v_0 \hat{x} \times (-\gamma B_z \hat{z})) = \gamma q v_0 B_z(-\hat{y} + \hat{y}) = 0$$

as expected.

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