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Consider a relativistic gas of N indistinguishable non-interacting spin-1/2 fermions of zero rest mass, initially at equilibrium in a (three-dimensional) volume V_i at zero temperature, $T_i = 0$. These are fictitious massless fermionic particles that have no antiparticles and have energy $\epsilon(\vec{p}) = c|\vec{p}|$, where $\vec{p}$ is the particle's momentum.

1 Part A

Calculate the initial total energy E_i of this zero-temperature relativistic Fermi gas.

For a relativistic quantum gas, the density of states is given by

$$g(E) = \frac{g_s V E}{2\pi^2 \hbar^3 c^3} \sqrt{E^2 - m^2 c^4} \tag{1}$$

where $g_s = 2s + 1$ is the degeneracy due to the particles being spin-s. Here $g_s = 2$, and $m = 0$. (See David Tong statistical physics notes chapter 3 for explanation of density of states.) If $n(E) = \frac{1}{e^{\beta(E-\mu)}+1}$ is the occupation number for fermions, then

$$N = \int_0^\infty g(E) n(E) dE \tag{2}$$

$$E = \int_0^\infty E g(E) n(E) dE \tag{3}$$

But at T=0, then $n(E)$ is either 0 or 1. Because we know N, we can find out the energy of the highest-energy filled state, E_F (known as the fermi energy).

$$N = \int_0^{E_F} g(E) dE = \frac{V E_F^3}{3\pi^2 c^3 \hbar^3} \tag{4}$$

Rearranging gives

$$E_F = c\hbar (\frac{3\pi^2 V}{N})^{1/3} \tag{5}$$

Thus the energy integral becomes

$$E = \int_0^{E_F} Eg(E)(1)dE = \frac{V}{4\pi^2 c^3 \hbar^3} E_F^4 \tag{6}$$

Plugging in our equation for E_F gives

$$E_i = \frac{c\hbar}{4}(3N)^{4/3}(\frac{\pi^2}{V_i})^{1/3} \tag{7}$$

2 Part B

The initial confining walls are then instantaneously removed and this gas expands into a vacuum to a much larger final volume V_f (enclosed by thermally insulating walls), and then internally equilibrates due to weak (and particle-number-conserving) interactions between the fermions. V_f is so large that quantum statistics can be ignored, and the final state of the gas can be treated as "classical", although still relativistic. What is the final temperature T_f of this gas?

Because the walls are thermally insulating, $dQ = 0$. Because the expansion is into a vacuum, $dW = 0$. Thus the change in internal energy of this gas is zero. Using

$$\langle E \rangle = -\frac{\partial}{\partial \beta} log(Z) \tag{8}$$

we can find the average energy of a classical gas. Remember that

$$Z_{classical} = \frac{1}{(2\pi\hbar)^3} \int \int d^3p d^3q e^{-\beta H(p,q)} \tag{9}$$

Here $H = pc$, the integral over q becomes V_f, and $d^3p = 4\pi p^2 dp$. Changing variables using $\beta pc = x$, we have

$$Z_{classical} = \frac{V}{2\pi^2 \hbar^3 c^3 \beta^3} \int_0^\infty x^2 e^{-x} dx \tag{10}$$

The integral equals 2. For N indistinguishable classical particles, $Z_N = Z_1^N/N!$, so $Z_N = \big(\frac{V_f k_B^3 T^3}{\pi^2 \hbar^3 c^3}\big)^N/N!$. Finally we have

$$\langle E_f \rangle = -\frac{\partial}{\partial \beta} \log Z_N = \frac{3N}{\beta} = 3Nk_B T \tag{11}$$

Setting $E_f = E_i$ gives us T_f, which is

$$T_f = \frac{c\hbar}{4k_B}\big(3\pi^2 \frac{N}{V_i}\big)^{1/3} \tag{12}$$

3 Part C

The initial entropy is 0, because there is only one way to arrange the system (all particles in the ground state). The final entropy is given by $\frac{\partial}{\partial T}(k_B T \log Z_N)$. Thus

$$\Delta S = S_f = Nk_B \log\big(\frac{V_f}{N\pi^2}(\frac{k_B T_f}{\hbar c})^3\big) + 4Nk_B = Nk_b\big[\log(\frac{3V_f}{64V_i}) + 4\big] \tag{13}$$