Consider a relativistic gas of N indistinguishable non-interacting spin-1/2 fermions of zero rest mass, initially at equilibrium in a (three-dimensional) volume $V_i$ at zero temperature, $T_i = 0$. These are fictitious massless fermionic particles that have no antiparticles and have energy $\epsilon(\vec{p}) = c|\vec{p}|$, where $\vec{p}$ is the particle’s momentum.

1 Part A

Calculate the initial total energy $E_i$ of this zero-temperature relativistic Fermi gas.

For a relativistic quantum gas, the density of states is given by

$$g(E) = \frac{g_s V E}{2\pi^2 \hbar^2 c^3} \sqrt{E^2 - m^2 c^4} \quad (1)$$

where $g_s = 2s + 1$ is the degeneracy due to the particles being spin-$s$. Here $g_s = 2$, and $m = 0$. (See David Tong statistical physics notes chapter 3 for explanation of density of states.) If $n(E) = \frac{1}{e^{\beta(E - \mu)} + 1}$ is the occupation number for fermions, then

$$N = \int_0^\infty g(E) n(E) dE \quad (2)$$

$$E = \int_0^\infty E g(E) n(E) dE \quad (3)$$

But at $T=0$, then $n(E)$ is either 0 or 1. Because we know $N$, we can find out the energy of the highest-energy filled state, $E_F$ (known as the fermi energy).

$$N = \int_0^{E_F} g(E) dE = \frac{V E_F^3}{3\pi^2 c^3 \hbar^3} \quad (4)$$

Rearranging gives

$$E_F = \frac{c \hbar}{N} \left(\frac{3\pi^2 V}{N}\right)^{1/3} \quad (5)$$
Thus the energy integral becomes
\[ E = \int_0^{E_F} E g(E)(1) dE = \frac{V}{4\pi^2 c^3 \hbar^3} E_F^4 \] (6)

Plugging in our equation for \( E_F \) gives
\[ E_i = \frac{\hbar c}{4} \frac{(3N)^{4/3}}{(\frac{\pi^2}{V_i})^{1/3}} \] (7)

2 Part B

The initial confining walls are then instantaneously removed and this gas expands into a vacuum to a much larger final volume \( V_f \) (enclosed by thermally insulating walls), and then internally equilibrates due to weak (and particle-number-conserving) interactions between the fermions. \( V_f \) is so large that quantum statistics can be ignored, and the final state of the gas can be treated as "classical", although still relativistic. What is the final temperature \( T_f \) of this gas?

Because the walls are thermally insulating, \( dQ = 0 \). Because the expansion is into a vacuum, \( dW = 0 \). Thus the change in internal energy of this gas is zero. Using
\[ \langle E \rangle = -\frac{\partial}{\partial \beta} \log(Z) \] (8)
we can find the average energy of a classical gas. Remember that
\[ Z_{\text{classical}} = \frac{1}{(2\pi \hbar)^d} \int \int d^3 p d^3 q e^{-\beta H(p,q)} \] (9)

Here \( H = pc \), the integral over \( q \) becomes \( V_f \), and \( d^3 p = 4\pi p^2 dp \). Changing variables using \( \beta pc = x \), we have
\[ Z_{\text{classical}} = \frac{V}{2\pi^2 \hbar^4 c^3 \beta^3} \int_0^\infty x^2 e^{-x} dx \] (10)

The integral equals 2. For \( N \) indistinguishable classical particles, \( Z_N = Z_1^N / N! \), so \( Z_N = \left( \frac{V_f k_B T_f}{\pi^2 \hbar^2 c^3} \right)^N / N! \). Finally we have
\[ \langle E_f \rangle = -\frac{\partial}{\partial \beta} \log Z_N = \frac{3N}{\beta} = 3N k_B T \] (11)

Setting \( E_f = E_i \) gives us \( T_f \), which is
\[ T_f = \frac{\hbar c}{4k_B} \left( \frac{3\pi^2 N}{V_f} \right)^{1/3} \] (12)
3 Part C

The initial entropy is 0, because there is only one way to arrange the system (all particles in the ground state). The final entropy is given by \( \frac{\partial}{\partial T} (k_B T \log Z_N) \). Thus

\[
\Delta S = S_f = N k_B \log \left( \frac{V_f}{N \pi^2} \left( \frac{k_B T_f}{\hbar c} \right)^3 \right) + 4Nk_B = N k_b \left[ \log \left( \frac{3V_f}{64V_i} \right) + 4 \right] \quad (13)
\]