J12T.3

Consider a gas with pressure

$$p(T,V) = aT^x$$

where a is a constant and the exponent x satisfies $x > 1$. Note that the equilibrium pressure of this gas does not depend on its volume V, only on its temperature T. Assume this gas has total energy $E(T, V = 0) = 0$ at $V = 0$ for all T, and has entropy $S(T = 0, V = 0) = 0$.

(a) What familiar textbook system could this be? What is the exponent x in that case? For general a and general $x > 1$, obtain the entropy $S(T,V)$ of this gas at equilibrium for all $T \geq 0$ and all $V \geq 0$.

Consider a reversible heat engine with this gas (for general a and general $x > 1$) as the working medium: Each cycle starts at volume V_A and temperature T_2. First isothermally expand the gas to volume V_B while in contact with the hot reservoir with temperature T_2. Then remove the gas from contact with the reservoirs and expand adiabatically until the temperature drops to T_1, the temperature of the cold reservoir ($T_1 < T_2$). Complete the cycle by compressing the gas first isothermally at T_1, then adiabatically, to return to the start of the next cycle.

(a) Sketch this cycle in the pV plane. Give the equations for $p(V)$ along all parts of the cycle. Make sure your sketch in the pV plane to qualitatively accurate.

(b) Show that the efficiency of this reversible heat engine is equal to the standard Carnot result.

Solution:

(a) This could describe the pressure of a photon gas which has an exponent of $x = 4$. Since we have a system of constant T and a pressure equation of state that id independent of V, we should use the Helmholtz free energy:

$$dF = -SdT - pdV$$

This means that

$$p = -\left(\frac{\partial F}{\partial V}\right)_T \qquad S = -\left(\frac{\partial F}{\partial T}\right)_V$$

From integrating the first equation, we find that the Helmoholtz free energy is

$$F = -aT^xV + g(T)$$

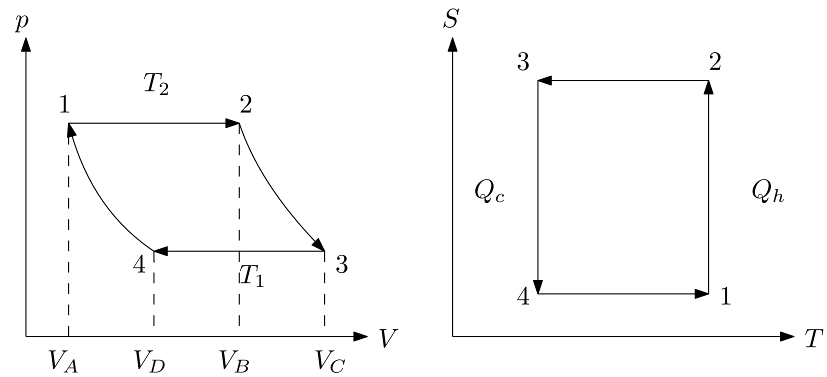
where $g(T)$ is a function of T. However, since F is an extensive quantity and T is intensive, we require that $g(T) = 0$ and therefore $F = -aT^xV$. Taking the negative derivative of F with respect to T at constant V yields the entropy of the system:

$$\boxed{S(V,T) = -\left(\frac{\partial F}{\partial T}\right)_V = axT^{x-1}V}$$

(b) We can also re-express S as a function of p and V:

$$S(p, V) = a^{1/x} x p^{1-\frac{1}{x}} V$$

Hence, for an adiabatic process, $\boxed{p^{1-\frac{1}{x}} V = A}$, where A is a constant. For an isothermal process, $\boxed{p = B}$, where B is constant, for all values of V. The pV curve and ST curve (which will be used in part (c)) looks as follows:



(c) From conservation of energy, $\Delta U = 0$ for the reversible cycle. Hence,

$$\Delta Q = -\Delta W \qquad \Rightarrow \qquad \oint T dS = \oint p dV$$

So instead of looking at the pV diagram, we can also draw on ST curve for the heat engine process to compute the work and efficiency of the process.

Note that during the adiabatic process, no heat is exchanged between the system and the reservoir. So, the only contributions to total heat exchange are from the isothermal expansion and compression. Note that

$$\Delta W = Q_H - Q_C$$

where Q_H is the heat transferred from the hot reservoir (at T_2) to the gas and Q_C is the heat transferred from the gas to the cold reservoir (at T_1). Hence, the efficiency can be rewritten as

$$\eta = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H}$$

From the entropy equation, we find that

$$Q_H = T_2 \Delta S = axT_2^x (V_B - V_A)$$
$$Q_C = T_1 \Delta S = axT_1^x (V_C - V_D)$$

Now, from the adiabatic equation, we know that $T^{x-1} V$ is constant. Therefore,

$$T_1^{x-1} V_D = T_2^{x-1} V_A \qquad \Rightarrow \qquad V_D = \left(\frac{T_2}{T_1}\right)^{x-1} V_A$$

$$T_1^{x-1} V_C = T_2^{x-1} V_B \qquad \Rightarrow \qquad V_C = \left(\frac{T_2}{T_1}\right)^{x-1} V_B$$

Substituting the volumes V_C and V_D into the equations for the heat exchanged, and these heat equations back into the efficiency, we find that

$$\eta = 1 - \frac{axT_1^x \left(\frac{T_2}{T_1}\right)^{x-1} (V_B - V_A)}{axT_2^x (V_B - V_A)} = 1 - \frac{T_1}{T_2}$$

which is exactly the efficiency of a standard Carnot cycle.

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