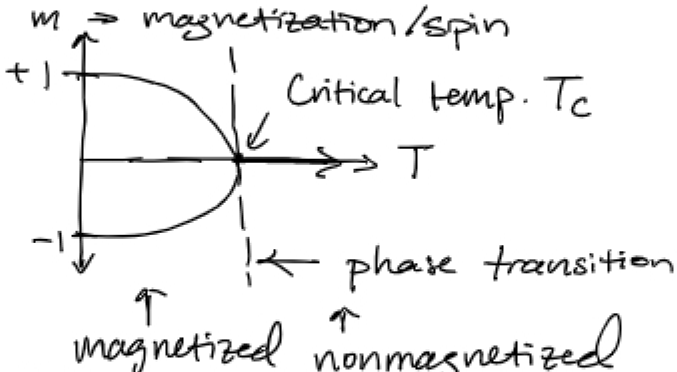
Consider N classical 2-state spins $S_i = \pm 1$, w/ Hamiltonian

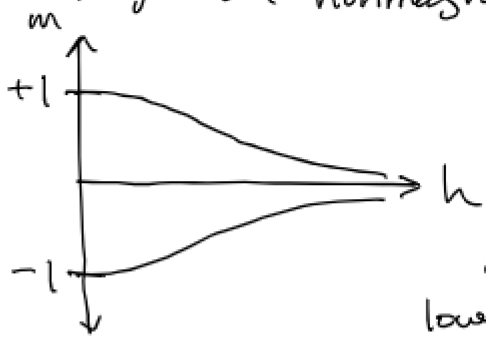
$$H = -\frac{J}{N}\sum_{i=1}^{N}\sum_{j=1}^{i-1} S_i S_j - h\sum_{i}^{N} S_i$$

This is an "infinite range" model where all spins interact with all others: the spin-spin coupling (J/N) is defined so that the energy J remains a positive constant in the thermodynamic limit $N \to \infty$. You should work in this limit. The external magnetic field h can be of either sign or be zero.

(a) Sketch the equilibrium phase diagram of this system vs. temperature T and field strength h, showing all phase transitions and critical points that occur as one varies T and/or h.



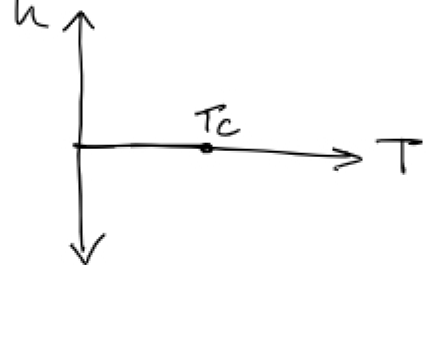
magnetization vs. temp for fixed h



magnetization vs. field strength

→ second order phase transition as T held constant & h is lowered/raised across 0

⇒ if h changes <u>sign</u>, we get phase transition since m also changes sign suddenly



Finally, field strength vs. temp.

if $h=0$ we get phase transition at $T=T_c$

⇒ T_c is a critical point

→ restatement of first plot @ $h=0$

(b) Calculate the critical temp T_c

Infinite range model ⇒ we can use mean field model in general, we replace interaction term (J) w/ overall magnetization of system

$$M = g\,J\,\langle s_i \rangle$$

\# of nearest neighbors — interaction coupling — ↳ average value of spin

all spins interact

for this model J is actually given by $\frac{J}{N}$, $g = N$

$$\Rightarrow M = J\langle s_i \rangle$$

$$\Rightarrow H = -M\sum_i s_i - h\sum_i s_i = (-J\langle s_i\rangle - h)\sum_i s_i$$

let $m = \langle s_i \rangle$

$$\Rightarrow H = (-Jm - h)\sum_i s_i$$

for a single spin site, $s_i = \pm 1 \Rightarrow \varepsilon_1 = -Jm - h$

$\varepsilon_2 = Jm + h$

then $Z_1 = e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2}$ for single particle

$$= e^{-\beta(-Jm-h)} + e^{-\beta(Jm+h)}$$

$$= e^{\beta Jm + \beta h} + e^{-\beta Jm - \beta h} = 2\cosh(\beta Jm + \beta h)$$

overall then

$$Z = Z_1^N = 2^N \cosh^N(\beta Jm + \beta h)$$

want to find $m = \langle s_i \rangle$

$$\langle s_i \rangle = \sum_i s_i\, p(s_i) = \sum \frac{s_i e^{-\beta E(s_i)}}{Z}$$

$$= \frac{+1\, e^{-\beta(-Jm-h)} + (-1)\, e^{-\beta(Jm+h)}}{e^{-\beta(-Jm-h)} + e^{-\beta(Jm+h)}} = \frac{e^{\beta Jm + \beta h} - e^{-\beta Jm - \beta h}}{2\cosh(\beta Jm + \beta h)}$$

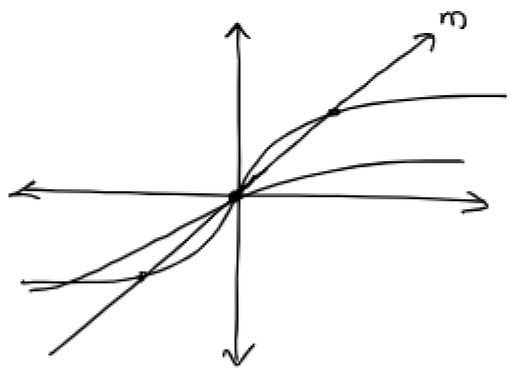
Z_1

$$= \frac{2\sinh(\beta J m + \beta h)}{2\cosh(\beta J m + \beta h)} = \tanh(\beta J m + \beta h)$$

could also have found $m = \langle s_i \rangle$ using $m = \frac{1}{\beta}\frac{\partial \log Z}{\partial h}$
& could get same result

$$m = \tanh(\beta J m + \beta h)$$

for $h = 0 \Rightarrow m = \tanh(\beta J m)$



Depending on the value of $\beta J m$, there may be only 1 solution ($m=0$) or 3 ($m = 0, \pm m_0$)

critical point when slope near origin of $\tanh(\beta J m) = m$

near origin $\beta J m \ll 1$ & Taylor expanding $\tanh(x)$

$\tanh(x) \simeq x \Rightarrow \tanh(\beta J m) \simeq \beta J m = m$ @ critical point

$$\Rightarrow \beta J = 1 \Rightarrow \beta = \frac{1}{J}$$

$$\frac{1}{k_B T_c} = \frac{1}{J}$$

$$\boxed{T_c = J/k_B}$$

(c) At $T = T_c$, calculate the magnetization $m = \langle s_i \rangle$ as a function of the field h in the limit where $|h|$ is small but not 0

At critical temp $\beta = 1/J$

$\Rightarrow m = \tanh(m + \beta h)$ now $h \neq 0$

Taylor expand $\tanh(x)$ near $x = 0$ again, but keep one more term

$$\tanh(x) \simeq x - \frac{x^3}{3}$$

$\Rightarrow \tanh(m + \beta h) \simeq m + \beta h - \frac{1}{3}(m + \beta h)^3 = m$ near origin

$$\Rightarrow \beta h - \frac{1}{3}(m+\beta h)^3 = 0$$

$$3\beta h = (m+\beta h)^3 \Rightarrow (3\beta h)^{1/3} = m+\beta h$$

$$m(h) = (3\beta h)^{1/3} - \beta h$$ at critical temp.

$$\Rightarrow \boxed{m(h) \simeq \left(\frac{3h}{J}\right)^{1/3} - \frac{h}{J}}$$ in terms of given variables