J12Q2

A system of two indistinguishable spin-1/2 particles is governed by the Hamiltonian:

\[ H = \frac{\mathbf{p}_1^2}{2m} + \frac{\mathbf{p}_2^2}{2m} + \lambda \frac{\mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2}{|\mathbf{x}_1 - \mathbf{x}_2|} \]

where \( \mathbf{\sigma} \) are the Pauli spin operators of the two particles, \( \mathbf{p} \) and \( \mathbf{x} \) are their 3-D momenta and positions.

Find the ground state energies for the two cases:
(a) \( \lambda > 0 \)
(b) \( \lambda < 0 \)
(c) State the degeneracies of the ground state in each case, in the center of mass frame.

Solution

*Here we assume \( \lambda \) is a dimensionless coefficient*

We take the hint and rewrite the Hamiltonian in CM frame, with relative momentum operator \( \mathbf{p} \), momentum of CM operator \( \mathbf{P} \) and relative position \( \mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2 \):

\[ H = \frac{\mathbf{p}^2}{2\mu} + \lambda \frac{\mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2}{|\mathbf{x}|} \]

where \( \mu \) is the reduced mass of the two particle system, which is \( m/2 \) here. As usual, we ignore the momentum of CM so the Hamiltonian that contributes to the system is simply:

\[ H = \frac{\mathbf{p}^2}{2\mu} + \lambda \frac{\mathbf{\sigma}_1 \cdot \mathbf{\sigma}_2}{|\mathbf{x}|} \]

This is then in the same form as the Hamiltonian for the hydrogen atom.

\[ H = \frac{\mathbf{p}^2}{2\mu} - \frac{e^2}{|\mathbf{x}|} \]

We can therefore borrow the form of the energy states:

\[ E_n = \frac{\mu e^4}{2\hbar^2 n^2} \]

The spin-spin coupling term has two eigenvalues \( \epsilon_1 = (-3)\lambda \frac{\hbar^2}{4} \), corresponding to the singlet state (anti-symmetric in particle exchange) and \( \epsilon_2 = \lambda \frac{\hbar^2}{4} \), corresponding to the triplet state (symmetric).

Substitute \( \epsilon_i \) for \( e^2 \) back into the hydrogen energy states will give us the energy spectrum.

(a)

When \( \lambda > 0 \), ground state is the spin singlet state, with \( n = 1, l = 0 \) position state. Ground state energy is therefore:

\[ E = \frac{\mu}{2\hbar^2} \epsilon_1 = \frac{9\lambda^2 \hbar^2}{32} = \frac{9\lambda^2 \hbar^2 m}{64} \]

Both position state and spin state are non-degenerate. Therefore this ground state is non-degenerate.

(b)

When \( \lambda < 0 \), the ground state is the spin triplet state. Because the triplet state is symmetric, we need an anti-symmetric position state. Therefore the position state is \( n = 2, l = 1 \) state.

\[ E = \frac{\mu}{2 \cdot 2 \cdot \hbar^2} \epsilon_2 = \frac{\mu \lambda^2 \hbar^2}{16 * 8} = \frac{\mu \lambda^2 \hbar^2}{16 * 16} \]

Both spin and position states have 3 fold degeneracy. Therefore this state has a 9-fold degeneracy.