J12Q2

A system of two indistinguishable spin-1/2 particles is governed by the hamiltonian:

$$H = \frac{\mathbf{p_1}^2}{2m} + \frac{\mathbf{p_2}^2}{2m} + \lambda \frac{\sigma_1 \cdot \sigma_2}{|\mathbf{x_1} - \mathbf{x_2}|}$$

where σ are the Pauli spin operators of the two particles, p and x are their 3-D momenta and positions. Find the ground state energies for the two cases:
(a)$\lambda > 0$
(b)$\lambda < 0$
(c) State the degeneracies of the ground state in each case, in the center of mass frame.

Solution

Here we assume λ is a dimensionless coefficient
W take the hint and rewrite the Hamiltonian in CM frame, with relative momentum operator $\mathbf{p}$, momentum of CM operator $\mathbf{P}$ and relative position $\mathbf{x} = \mathbf{x_1} - \mathbf{x_2}$:

$$H = \frac{\mathbf{P}^2}{2\mu} + \frac{\mathbf{p}^2}{2\mu} + \lambda \frac{\sigma_1 \cdot \sigma_2}{|\mathbf{x}|}$$

where μ is the reduced mass of the two particle system, which is $m/2$ here. As usual, we ignore the momentum of CM so the Hamiltonian that contributes to the system is simply:

$$H = \frac{\mathbf{p}^2}{2\mu} + \lambda \frac{\sigma_1 \cdot \sigma_2}{|\mathbf{x}|}$$

This is then in the same form as the Hamiltonian for the hydrogen atom.

$$H = \frac{\mathbf{p}^2}{2\mu} - \frac{e^2}{|\mathbf{x}|}$$

We can therefore borrow the form of the energy states:

$$E_n = \frac{\mu e^4}{2\hbar^2 n^2}$$

The spin-spin coupling term has two eigenvalues $\epsilon_1 = (-3)\lambda\frac{\hbar^2}{4}$, corresponding to the singlet state (anti-symmetric in particle exchange) and $\epsilon_2 = \lambda\frac{\hbar^2}{4}$, corresponding to the triplet state (symmetric).
Substitute ϵ_i for e^2 back into the hydrogen energy states will give us the energy spectrum.

(a)

When $\lambda > 0$, ground state is the spin singlet state, with $n = 1, l = 0$ position state. Ground state energy is therefore:

$$E = \frac{\mu}{2\hbar^2}\epsilon_1^2 = \frac{9\lambda^2\hbar^2\mu}{32} = \frac{9\lambda^2\hbar^2 m}{64}$$

Both position state and spin state are non-degenerate. Therefore this ground state is non-degenerate.

(b)

When $\lambda < 0$, the ground state is the spin triplet state. Because the triplet state is symmetric, we need an anti-symmetric position state. Therefore the position state is $n = 2, l = 1$ state.

$$E = \frac{\mu}{2 \cdot 2^2\hbar^2}\epsilon_2^2 = \frac{\mu\lambda^2\hbar^2}{16 * 8} = \frac{\mu\lambda^2\hbar^2}{16 * 16}$$

Both spin and position states have 3 fold degeneracy. Therefore this state has a 9-fold degeneracy.