January 2012 Preliminary Exam, Quantum Mechanics Problem 1

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Problem:

Consider a two-level system with two orthogonal states $|g\rangle$ and $|e\rangle$. It has the time-dependent Hamiltonian:

$$H(t) = \hbar\omega |e\rangle \langle e| + V\cos(\omega t)(|e\rangle \langle g| + |g\rangle \langle e|) \tag{1}$$

Assume that the time-dependent term is small, $\hbar\omega \gg V > 0$, so that you may make the corresponding approximation. At time $t = 0$ the state of the system is specified by the initial complex amplitude c_{g0} and c_{eo}:

$$|\psi(t = 0)\rangle = c_{g0} |g\rangle + c_{eo} |e\rangle \tag{2}$$

What is the state of the system $|\psi(t)\rangle$ at other times t?

Reason for New Solution:

To correct sign errors in derivation, and to elaborate on justifications of solution steps.

Solution:

We know that, at later times, the wave function will have the following form:

$$|\psi(t)\rangle = c_g(t) |g\rangle + c_e(t) |e\rangle \tag{3}$$

Thus, our job is to solve for these time-dependent coefficients, which are also the transition amplitudes to their respective initial states. We will treat this problem in the "interaction picture", where the diagonal element of $H(t)$ will be called H_0, and the off-diagonal, perturbing elements of $H(t)$ will be called H_1, both in the Schrodinger picture. I will also use an additional subscript I to indicate operators in the interaction picture. Thus, to first order,

$$|\psi(t)\rangle = W(t) |\psi(0)\rangle \tag{4}$$

where

$$W(t) \approx 1 + \frac{1}{i\hbar} \int_0^t dt' H_{1I}(t') \tag{5}$$

We convert H_{1I}, in the interaction picture, to H_1, in the Schrodinger picture, using the following unitary transformation:

$$H_{1I}(t) = e^{iH_0 t} H_1(t) e^{-iH_0 t} \tag{6}$$

Therefore,

$$
\begin{aligned}
c_g(t) &= \langle g|\, W(t)\, |\psi(0)\rangle = \langle g|\, 1 + \frac{1}{i\hbar}\int_0^t dt' H_{1I}(t')(c_{go}\,|g\rangle + c_{eo}\,|e\rangle) \\
&= c_{go} + \frac{V}{i\hbar}\int_0^t dt'\, \langle g|\, e^{iH_0 t'}[\cos(\omega t')(|e\rangle\,\langle g| + |g\rangle\,\langle e|)]e^{-iH_0 t'}(c_{go}\,|g\rangle + c_{eo}\,|e\rangle) \\
&= c_{go} + \frac{Vc_{eo}}{i\hbar}\int_0^t dt'\, \langle g|\, e^{iH_0 t'}[\cos(\omega t')\,|g\rangle\,\langle e|]e^{-iH_0 t'}\,|e\rangle \\
&= c_{go} + \frac{Vc_{eo}}{i\hbar}\int_0^t dt'\, \langle g|\, [\cos(\omega t')\,|g\rangle\,\langle e|]e^{-i\omega t'}\,|e\rangle \\
&= c_{go} + \frac{Vc_{eo}}{i\hbar}\int_0^t dt' \cos(\omega t')e^{-i\omega t'} = c_{go} + \frac{Vc_{eo}}{2i\hbar}\int_0^t dt'(e^{i\omega t'} + e^{-i\omega t'})e^{-i\omega t'} \\
&= c_{go} + \frac{Vc_{eo}}{2i\hbar}\left[t - \frac{1}{2i\omega}(e^{-2i\omega t'} - 1)\right]
\end{aligned}
\tag{7}
$$

Similarly,

$$
\begin{aligned}
c_e(t) &= \langle e|\, W(t)\, |\psi(0)\rangle = \langle e|\, 1 + \frac{1}{i\hbar}\int_0^t dt' H_{1I}(t')(c_{go}\,|g\rangle + c_{eo}\,|e\rangle) \\
&= c_{eo} + \frac{V}{i\hbar}\int_0^t dt'\, \langle e|\, e^{iH_0 t'}[\cos(\omega t')(|e\rangle\,\langle g| + |g\rangle\,\langle e|)]e^{-iH_0 t'}(c_{go}\,|g\rangle + c_{eo}\,|e\rangle) \\
&= c_{eo} + \frac{Vc_{go}}{i\hbar}\int_0^t dt'\, \langle e|\, e^{iH_0 t'}[\cos(\omega t')\,|e\rangle\,\langle g|]e^{-iH_0 t'}\,|g\rangle \\
&= c_{eo} + \frac{Vc_{go}}{i\hbar}\int_0^t dt'\, \langle e|\, e^{i\omega t'}[\cos(\omega t')\,|e\rangle\,\langle g|]\,|g\rangle \\
&= c_{eo} + \frac{Vc_{go}}{i\hbar}\int_0^t dt' \cos(\omega t')e^{i\omega t'} = c_{eo} + \frac{Vc_{go}}{2i\hbar}\int_0^t dt'(e^{i\omega t'} + e^{-i\omega t'})e^{i\omega t'} \\
&= c_{eo} + \frac{Vc_{go}}{2i\hbar}\left[t + \frac{1}{2i\omega}(e^{2i\omega t'} - 1)\right]
\end{aligned}
\tag{8}
$$

To get the final solution (the wave function at time t), we substitute these coefficients into (3).