A particle of charge $Q$ and mass $m$ is constrained to move along a circle of radius $R$, which lies in the $z = 0$ plane, centered at $(x, y, z) = (0, 0, 0)$. Passing through this circle is an ideal infinitely long straight cylindrical solenoid of radius $R_0 < R$, which is centered on the $z$ axis. Assume that the magnetic field vanishes outside of the solenoid, and is constant, $\vec{B} = B\hat{z}$.

(a) Write down a Hamiltonian for this particle, in terms of the angular position $\theta$ of the particle on the circle, and the given parameters $B$, $Q$, $m$, $R$ and $R_0$.

(b) Find the complete spectrum of this particle’s eigenenergies, as a function of the given parameters.

Solution:

(a) The electromagnetic Hamiltonian with no electric potential is given by

$$H = \frac{1}{2m} (\vec{p} - Q\vec{A})^2$$

where $\vec{A}$ is the magnetic vector potential. For a cylindrical solenoid, the magnetic field is confined to the region inside the cylinder, and has constant field $\vec{B} = B\hat{z}$. Since $\vec{B} = \nabla \times \vec{A}$, then the vector potential outside the solenoid is

$$A_\phi (2\pi r) = B(\pi R_0^2) \quad \Rightarrow \quad A_\phi = \frac{BR_0^2}{2r}$$

For a particle confined on a ring of radius $R$, the Hamiltonian is

$$H = \frac{1}{2m} \left( \frac{\hbar}{iR} \frac{\partial}{\partial \phi} - \frac{QBR_0^2}{2R} \right)^2$$

(b) From time-independent Schrödinger’s equation,

$$\frac{1}{2m} \left( -\frac{\hbar^2}{R^2} \frac{\partial^2}{\partial \phi^2} + \left( \frac{QBR_0^2}{2R} \right)^2 - 2 \left( \frac{QBR_0^2}{2R} \right) \left( \frac{\hbar}{iR} \right) \frac{\partial}{\partial \phi} \right) \psi(\phi) = E \psi(\phi)$$

This Hamiltonian is very similar to a free particle confined to a ring, so we will guess solutions of $\psi(\phi) = e^{in\phi}$, where $n$ is an integer to satisfy periodic boundary conditions $\psi(\phi) = \psi(\phi + 2\pi)$. Substituting this into the above equation, we get the energy spectrum

$$\frac{\hbar^2}{R^2} n^2 + \left( \frac{QBR_0^2}{2R} \right)^2 - 2 \left( \frac{QBR_0^2}{2R} \right) \left( \frac{\hbar}{iR} \right) n = 2mE_n$$

$$\left( \frac{\hbar}{R} n - \frac{QBR_0^2}{2R} \right)^2 = 2mE_n$$

$$\frac{\hbar^2}{2mR^2} \left( n - \frac{QBR_0^2}{2\hbar} \right)^2 = E_n$$

$\blacksquare$