J12Q.3 (Solution by Jim Wu)

A particle of charge Q and mass m is constrained to move along a circle of radius R, which lies in the $z = 0$ plane, centered at $(x, y, z) = (0, 0, 0)$. Passing through this circle is an ideal infinitely long straight cylindrical solenoid of radius $R_0 < R$, which is centered on the z axis. Assume that the magnetic field vanishes outside of the solenoid, and is constant, $\vec{B} = B\hat{z}$.

(a) Write down a Hamiltonian for this particle, in terms of the angular position θ of the particle on the circle, and the given parameters B, Q, m, R and R_0.

(b) Find the complete spectrum of this particle's eigenenergies, as a function of the given parameters.

Solution:

(a) The electromagnetic Hamiltonian with no electric potential is given by

$$H = \frac{1}{2m}(\vec{p} - Q\vec{A})^2$$

where $\vec{A}$ is the magnetic vector potential. For a cylindrical solenoid, the magnetic field is confined to the region inside the cylinder, and has constant field $\vec{B} = B\hat{z}$. Since $\vec{B} = \nabla \times \vec{A}$, then the vector potential outside the solenoid is

$$A_\phi(2\pi r) = B(\pi R_0^2) \qquad \Rightarrow \qquad A_\phi = \frac{BR_0^2}{2r}$$

For a particle confined on a ring of radius R, the Hamiltonian is

$$H = \frac{1}{2m}\left(\frac{\hbar}{iR}\frac{\partial}{\partial\phi} - \frac{QBR_0^2}{2R}\right)^2$$

(b) From time-independent Schrödinger's equation,

$$\frac{1}{2m}\left(-\frac{\hbar^2}{R^2}\frac{\partial^2}{\partial\phi^2} + \left(\frac{QBR_0^2}{2R}\right)^2 - 2\left(\frac{QBR_0^2}{2R}\right)\left(\frac{\hbar}{iR}\right)\frac{\partial}{\partial\phi}\right)\psi(\phi) = E\psi(\phi)$$

This Hamiltonian is very similar to a free particle confined to a ring, so we will guess solutions of $\psi(\phi) = e^{in\phi}$, where n is an integer to satisfy periodic boundary conditions $\psi(\phi) = \psi(\phi + 2\pi)$. Substituting this into the above equation, we get the energy spectrum

$$\begin{aligned}
\frac{\hbar^2}{R^2}n^2 + \left(\frac{QBR_0^2}{2R}\right)^2 - 2\left(\frac{QBR_0^2}{2R}\right)\left(\frac{\hbar}{R}\right)n &= 2mE_n \\
\left(\frac{\hbar}{R}n - \frac{QBR_0^2}{2R}\right)^2 &= 2mE_n \\
\frac{\hbar^2}{2mR^2}\left(n - \frac{QBR_0^2}{2\hbar}\right)^2 &= E_n
\end{aligned}$$

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