J12M1 (Solution by Jim Wu)
A thin ring of radius R, total mass M, and uniform mass density is placed on a horizontal frictionless surface. An ant of mass m with negligible moment of inertia is placed on the ring. The ant walks around the ring, completing one full circle relative to the ring. Relative to the frictionless surface, through what angle ψ does the ring turn.

Solution:
Since there are no external forces on the system, then the center of mass, located at a distance

$$r_{cm} = \frac{mR}{M+m} \tag{1}$$

between the center of the circle (origin) and the ant, must be stationary. Therefore, angular momentum is conserved around the center of mass. For convenience, let's work in a fixed coordinate system with the origin at the center of mass.

First, from a stationary center of mass, we have

$$M\mathbf{r}_{ring} + m\mathbf{r}_{bug} = 0 \tag{2}$$

where $\mathbf{r}_{ring}$ and $\mathbf{r}_{bug}$ are the positions of the ring's center and bug, respectively, to our fixed coordinate system. But we also require that the bug stay on the ring the entire time, implying that

$$\mathbf{R} = \mathbf{r}_{bug} - \mathbf{r}_{ring} \tag{3}$$

where $\mathbf{R}$ is the position of the bug relative to the ring's center as seen in the fixed frame, which is equal in magnitude to the radius of the circle. Solving this system of equations, we find that

$$\mathbf{r}_{bug} = \frac{M\mathbf{R}}{m+M} \qquad \mathbf{r}_{ring} = -\frac{m\mathbf{R}}{m+M} \tag{4}$$

The above equations suggest that the bug and the ring's center are collinear with the center of mass, and they orbit around the center of mass in opposite directions and with different radii, as expected.

Since the ring and the ant were originally at rest, the total angular momentum must be zero at all times. Without loss of generality, let's arbitrarily say that the bug moves in the counterclockwise direction, which immediately implies that the ring must rotate about the center of mass in the clockwise direction. The ring's angular momentum is

$$\mathbf{L}_{ring} = (I_{cm} + M\mathbf{r}_{ring}^2)\boldsymbol{\omega}_{ring} = -MR^2\left(1 + \frac{m^2}{(m+M)^2}\right)\omega_{ring}\hat{\mathbf{z}} \tag{5}$$

Now, the velocity of the bug $\mathbf{v}_{bug}$ seems to be slightly more tricky because the bug is moving at constant speed relative to the ring and the ring is also moving under it's feet. Recall that the relation between the rate of change of a vector $\mathbf{A}$ observed in the inertial frame and the non-inertial frame is

$$\left(\frac{d\mathbf{A}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{A}}{dt}\right)_{noninertial} + \boldsymbol{\Omega}\times\mathbf{A}$$

where $\mathbf{\Omega}$ is the angular velocity of non-inertial observer relative to the inertial frame. Applying this to $\mathbf{r}_{bug}$, we have

$$\begin{aligned}\left(\frac{d\mathbf{r}_{bug}}{dt}\right)_{inertial} &= \left(\frac{d\mathbf{r}_{bug}}{dt}\right)_{non-inertial} + \boldsymbol{\omega}_{ring} \times \mathbf{r}_{bug} \\ &= \mathbf{v}_{rel} + \frac{M}{m+M}\boldsymbol{\omega}_{ring} \times \mathbf{R}\end{aligned}$$

where $\mathbf{v}_{rel}$ is the velocity of the bug relative to the ring's center in the frame of the ring. Note that $\mathbf{v}_{rel}$ is in a direction tangential to the ring and since $\boldsymbol{\omega}_{ring}$ is in the $-\hat{\mathbf{z}}$ direction, then $\boldsymbol{\omega}_{ring} \times \mathbf{R}$ is also in the same direction as $\mathbf{v}_{rel}$. As expected the bug's velocity in the inertial frame is boosted by the ring's counter-rotation!

The angular momentum of the bug relative to the center of mass is

$$\mathbf{L}_{bug} = m\mathbf{r}_{bug} \times \left(\frac{d\mathbf{r}_{bug}}{dt}\right)_{inertial} = \frac{mM}{m+M}\mathbf{R} \times \left(\mathbf{v}_{rel} + \frac{M}{m+M}\boldsymbol{\omega}_{ring} \times \mathbf{R}\right)$$

Note that $\mathbf{R}$ is perpendicular to both $\mathbf{v}_{rel}$ and $\boldsymbol{\omega}_{ring} \times \mathbf{R}$, and so

$$\mathbf{L}_{bug} = \left(\frac{mM}{m+M}Rv + \frac{mM^2}{(m+M)^2}R^2\omega_{ring}\right)\hat{\mathbf{z}}$$

Therefore, from conservation of angular momentum,

$$\begin{aligned}\frac{mM}{m+M}Rv + \frac{mM^2}{(m+M)^2}R^2\omega_{ring} &= MR^2\left(1 + \frac{m^2}{(m+M)^2}\right)\omega_{ring} \\ (-mM + (m+M)^2 + m^2)\omega_{ring} &= (m+M)m\frac{v}{R} \\ \omega_{ring} &= \frac{(m+M)m}{M^2+Mm+m^2}\frac{v}{R} \\ \frac{d\psi}{dt} &= \frac{(m+M)m}{M^2+Mm+m^2}\frac{ds}{dt}\end{aligned}$$

where s is the distance traversed by the ant along the ring. If the ant makes one trip around the ring of distance $2\pi R$, then the ring has rotated an angle

$$\psi = 2\pi\frac{(m+M)m}{M^2+Mm+m^2}$$

relative to the frictionless surface. Notice that $\psi < 2\pi$, which makes sense as the counter-rotation of the ring assists the ant in making one round trip faster. Furthermore, if $M \gg m$, then the denominator is much larger and the ring does not rotate, which also agrees with out intuition.

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