j12e2

$$\omega_p^2 = \frac{ne^2}{m\varepsilon_0}.$$

First, find plasma dispersion relation (spoiler: $\omega^2 = c^2k^2 + \omega_p^2$)

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0} = 0 \qquad \nabla \cdot B = 0$$

$$\nabla \times E = -\partial B/\partial t \qquad \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}.$$

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = -\frac{\partial}{\partial t} \nabla \times B = -\mu_0 \frac{\partial J}{\partial t} - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}.$$

$n = n_0 + n_1$ all quantities $f_1 \sim e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$V = V_0 + V_1$ $V_0 = 0$ also $J = -enV = -en_0 V_1$

$E = E_0 + E_1$ $E_0 = 0$ and note $m\frac{dV_1}{dt} = -eE_1$

$$\nabla(0) - (ik)^2 E_1 = -\mu_0(-en_0)\left(\frac{-e}{m}E_1\right) - \mu_0\varepsilon_0(-i\omega)^2 E_1$$

$$k^2 = -\frac{\mu_0 e^2 n_0}{m} + \mu_0 \varepsilon_0 \omega^2$$

$$\omega^2 = k^2c^2 + \omega_p^2 \quad \text{as expected.}$$

We want to compare group velocities. $V_g = \frac{d\omega}{dk}$.

$$V_g: \quad 2\omega\frac{d\omega}{dk} = 2kc^2 \rightarrow V_g = \frac{c^2 k}{\omega}$$

$$\text{so} \quad V_{g1} - V_{g2} = c^2\left(\frac{k_1}{\omega_1} - \frac{k_2}{\omega_2}\right) = c^2\left[\frac{1}{c}\sqrt{1-\left(\frac{\omega_p}{\omega_1}\right)^2} - \frac{1}{c}\sqrt{1-\left(\frac{\omega_p}{\omega_2}\right)^2}\right]$$

$$V_{g1} - V_{g2} = c\left(1 - \frac{1}{2}\left(\frac{\omega_p}{\omega_1}\right)^2 - \left[1 - \frac{1}{2}\left(\frac{\omega_p}{\omega_2}\right)^2\right]\right) = \frac{c\omega_p^2}{2}\left(\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}\right)$$

Thus $\Delta T = \frac{D}{\Delta V}$ ← distance of source from earth

$$\Delta T = \frac{2D}{c\omega_p^2}\left(\frac{1}{\omega_2^2} - \frac{1}{\omega_1^2}\right)^{-1}.$$

since $\omega_2 > \omega_1$, this quantity is negative, thus $V_{g2} > V_{g1}$, so the pulse with ω_2 arrives first

$$= \frac{2D}{c\omega_p^2}\left(\frac{\omega_1^2\omega_2^2}{\omega_1^2 - \omega_2^2}\right)$$

$$= \frac{2D}{c\omega_p^2}\left(\frac{\omega_1^2}{(\omega_1/\omega_2)^2 - 1}\right) \approx \frac{2D\omega^2}{c\,\omega_p^2} \text{ in magnitude}$$

(using the fact that $\omega_1 \approx \omega_2$)