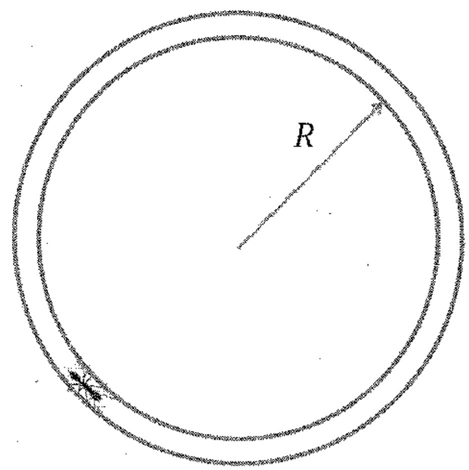
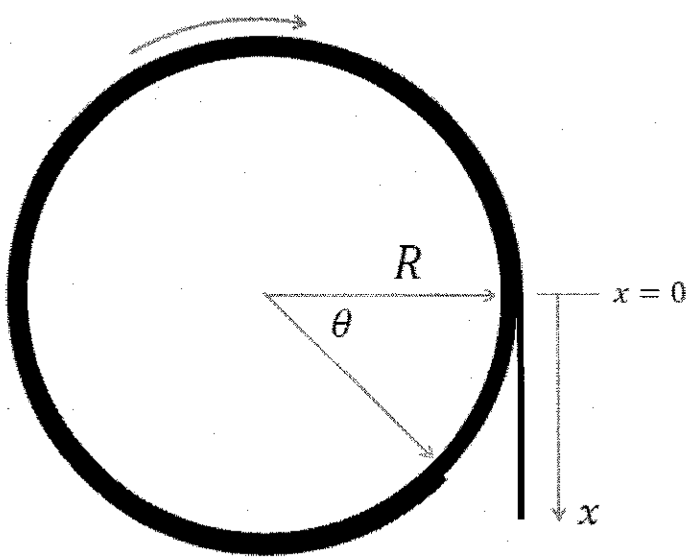
Section A. Mechanics

1. A thin ring of radius R, total mass M, and uniform mass density is placed on a horizontal frictionless surface. An ant of mass m with negligible moment of inertia is placed on the ring. The ant walks around the ring, completing one full circle relative to the ring. Relative to the frictionless surface, through what angle ψ does the ring turn?



2. A paper towel roll consists of thin towels with total mass m, linear mass density λ, and negligible thickness, wrapped around a thin hollow tube of mass M and radius R. The tube's central axis, about which it rotates without friction, is horizontal. The free end of the first towel is at $x = 0$ initially and the corresponding point on the tube is at $\theta = 0$. Initially, the towels wrap a precise integer number of times around the tube so that the mass density is uniform around the tube. Then the roll is rotated clockwise by an arbitrarily small angle so that a short segment of towel separates from the roll (see figure) and the roll is thus no longer in mechanical equilibrium. The roll then starts to rotate clockwise from rest as the towels unwind from the roll. Assume that the unwound towels are exactly vertical and never touch another surface. Also assume that the towels do not slip with respect to each other or with respect to the tube, and neglect any dissipation anywhere in this system.

What is the angular velocity of the roll as a function of θ?



3. An upright cylindrical bucket has radius R and its rim is at height H. This bucket is placed on a horizontal surface and filled to a height $h < H$ with incompressible water. The bucket is then rotated at angular frequency ω about a vertical axis that goes through the center of the bucket. Let g be the acceleration due to gravity, and assume that the water is simply rotating with the same angular velocity as the bucket.

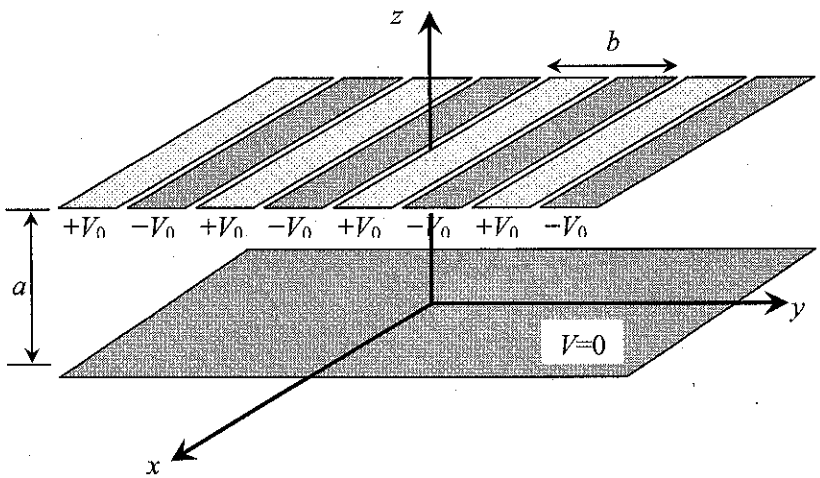
   a) Assuming that ω is small enough that the water does not reach the rim of the bucket or reveal the bottom of the bucket, find an expression for the height of the water's surface as a function of the radial distance from the central axis of rotation.

   b) Quantitatively, what are the conditions on ω such that water neither spills over the rim of the bucket nor reveals the bottom of the bucket?

Section B. Electricity and Magnetism

1. A device consists of two large (say, infinite) parallel planes. The bottom plane, at $z = 0$, is at voltage $V = 0$. The top plane, at $z = a$, is made of strips parallel to the x-axis of width $b/2$. They alternate in voltage between $+V_0$ and $-V_0$ as shown in the picture. (Ignore the gaps between the strips.)

   Find the electrostatic potential $V(x, y, z)$ everywhere between the planes.



2. A radio source (say a pulsar), a distance d from Earth (typically hundreds of light years), simultaneously emits two radio pulses, one at a frequency ω_1 and the other at ω_2, with ω_2 larger than ω_1 by a little. If the interstellar medium contains N free electrons per unit volume (and the same number density of free protons), what is the difference in arrival times, at the Earth, of the two pulses? Which one arrives earlier? You should assume that ω_1, $\omega_2 \gg \omega_p$, where ω_p is the plasma frequency of the medium.

3. An uncharged particle of mass M and magnetic moment m sits in a vacuum above a superconductor. The surface of the superconductor is the infinite plane $z = 0$. It is an ideal superconductor, so the magnetic field vanishes ($\vec{B} = 0$) inside the superconductor ($z < 0$). The particle's position and the orientation of its magnetic moment are those that minimize its energy in the presence of gravity $\vec{g} = -g\hat{z}$.

   (a) How far above the superconductor does the particle sit?

   (b) What is the orientation of its magnetic moment relative to the z-axis?

Section A. Quantum Mechanics

1. Consider a two-level system with two orthogonal states $|g\rangle$ and $|e\rangle$. It has the time-dependent Hamiltonian:

$$H(t) = \hbar\omega|e\rangle\langle e| + V\cos(\omega t)(|e\rangle\langle g| + |g\rangle\langle e|) \ .$$

Assume that the time-dependent term is small, $\hbar\omega \gg V > 0$, so that you may make the corresponding approximation. At time $t = 0$ the state of the system is specified by the initial complex amplitudes c_{g0} and c_{e0}:

$$|\psi(t=0)\rangle = c_{g0}|g\rangle + c_{e0}|e\rangle \ .$$

What is the state of the system $|\psi(t)\rangle$ at other times t?

2. A system of two indistinguishable spin-1/2 particles is governed by the Hamiltonian

$$H = \frac{|\vec{p}_1|^2}{2m} + \frac{|\vec{p}_2|^2}{2m} + \lambda \frac{\vec{\sigma}_1 \cdot \vec{\sigma}_2}{|\vec{x}_1 - \vec{x}_2|} ,$$

where $\vec{\sigma}_k$ $(k = 1, 2)$ are the Pauli spin operators of the two particles, and $\vec{p}_k$, $\vec{x}_k$ are their (3-dimensional) momenta and positions, respectively. Find the ground state energies for the two cases:

(a) $\lambda > 0$.

(b) $\lambda < 0$.

(c) For each sign of λ specify also the degeneracy of the ground state, in the center of mass frame.

3. A particle of charge Q and mass m is constrained to move along a circle of radius R, which lies in the $z = 0$ plane, centered at $(x, y, z) = (0, 0, 0)$. Passing through this circle is an ideal infinitely long straight cylindrical solenoid of radius $R_0 < R$, which is centered on the z axis. Assume that the magnetic field vanishes outside of the solenoid, and is constant, $\vec{B} = B\hat{z}$, inside the solenoid.

   (a) Write down a Hamiltonian for this particle, in terms of the angular position θ of the particle on the circle, and the given parameters B, Q, m, R and R_0.

   (b) Find the complete spectrum of this particle's eigenenergies, as a function of the given parameters.

Section B. Statistical Mechanics and Thermodynamics

1. Consider a relativistic gas of N indistinguishable non-interacting spin-1/2 fermions of zero rest mass, initially at equilibrium in a (three-dimensional) volume V_i at zero temperature, $T_i = 0$. These are fictitious massless fermionic particles that have no antiparticles and have energy $\epsilon(\vec{p}) = c|\vec{p}|$, where $\vec{p}$ is the particle's momentum.

   a) Calculate the initial total energy E_i of this zero-temperature relativistic Fermi gas.

   b) The initial confining walls are then instantaneously removed and this gas expands into a vacuum to a much larger final volume V_f (enclosed by thermally insulating walls), and then internally equilibrates due to weak (and particle-number-conserving) interactions between the fermions. V_f is so large that quantum statistics can be ignored, and the final state of the gas can be treated as "classical", although still relativistic. What is the final temperature T_f of this gas?

   c) What was the change in the entropy ΔS of the gas due to this expansion?

2. Consider N classical two-state spins $S_i = \pm 1$, with Hamiltonian

$$H = -\frac{J}{N}\sum_{i=1}^{N}\sum_{j=1}^{i-1} S_i S_j - h\sum_{i}^{N} S_i \ .$$

This is an "infinite-range" model where all spins interact with all others: the spin-spin coupling (J/N) is defined so the energy J remains a positive constant in the thermodynamic limit $N \to \infty$. You should work in this limit. The external magnetic field h can be of either sign or be zero.

a) Sketch the equilibrium phase diagram of this system vs. temperature T and field h, showing all phase transitions and critical points that occur as one varies T and/or h.

b) Calculate the critical temperature T_c.

c) At the critical temperature $T = T_c$, calculate the the magnetization $m = \langle S_i \rangle$ as a function of the field h in the limit where $|h|$ is small but nonzero.

3. Consider a gas with pressure

$$p(T, V) = aT^x ,$$

where a is a constant and the exponent x satisfies $x > 1$. Note that the equilibrium pressure of this gas does not depend on its volume V, only on its temperature T. Assume this gas has total energy $E(T, V = 0) = 0$ at $V = 0$ for all T, and has entropy $S(T = 0, V = 0) = 0$.

(a) What familiar textbook system could this be? What is the exponent x in that case? For general a and general $x > 1$, obtain the entropy $S(T, V)$ of this gas at equilibrium for all $T \geq 0$ and all $V \geq 0$.

Consider a reversible heat engine with this gas (for general a and general $x > 1$) as the working medium: Each cycle starts at volume V_A and temperature T_2. First isothermally expand the gas to volume V_B while in contact with the hot reservoir with temperature T_2. Then remove the gas from contact with the reservoirs and expand adiabatically until the temperature drops to T_1, the temperature of the cold reservoir ($T_1 < T_2$). Complete the cycle by compressing the gas first isothermally at T_1, then adiabatically, to return to the start of the next cycle.

(b) Sketch this cycle in the pV plane. Give the equations for $p(V)$ along all parts of the cycle. Make sure your sketch in the pV plane is qualitatively accurate.

(c) Show that the efficiency of this reversible heat engine is equal to the standard Carnot result.