J11Q.2
A particle of mass m is confined to the interval $[0, L]$ by a one-dimensional infinite square well. It is initially in the ground state of the Hamiltonian with the confining potential.

(a) At time $t = 0$ the potential within the well is suddenly changed to

$$V(x) = \begin{cases} V_0 & \text{for } 0 < x < L/2 \\ 0 & \text{for } L/2 < x < L \end{cases}$$

with $V_0 \ll E_1 - E_0$ (the latter being the gap between the two lowest states of the initial operator).

The perturbation lasts for time T, after which the potential is restored to its initial value. What is the probability that after the potential is restored the particle's energy is E_1, calculated to first order in $V_0/(E_1 - E_0)$?

(b) In a second experiment the value of V_0 (in the perturbing potential, as above) is increased very slowly, and to a much higher value $\overline{V} \gg E_1$.It is switched off instantaneously when that value is reached. What is the probability that at this point the particle will have energy E_1?

Solution:

(a) First, I would like to say that this problem uses horrible notation. For a particle in a box, the quantum number should start at $n = 1$, not zero! I refuse to use this lousy notation and from henceforth, I will use the correct labeling, i.e. E_1 for the ground state, and E_2 for the first excited state.

End rant....

For a particle in a box the energy levels and the corresponding eigenstates are

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \qquad \psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

where $n = 1, 2, 3, \ldots$

The perturbing potential, with V much smaller than the energy gap $E_2 - E_1$, is turned on at time $t = 0$ and turned off at time T. From time-dependent perturbation theory, the probability amplitude for ground state to transition to the excited state after a time T is

$$a_{1 \to 2}(T) = -\frac{i}{\hbar} \int_0^T \langle \psi_2 | V(x) | \psi_1 \rangle e^{i(E_2 - E_1)t/\hbar} \, dt$$

Note that the perturbation is from $0 < x < L/2$, and so

$$\begin{aligned}
\langle \psi_2 | V(x) | \psi_1(x) \rangle &= \frac{2V_0}{L} \int_0^{L/2} \sin \frac{2\pi x}{L} \sin \frac{\pi x}{L} \, dx \\
&= \frac{4V_0}{L} \int_0^{L/2} \sin^2 \frac{\pi x}{L} \cos \frac{\pi x}{L} \, dx \\
&= \frac{4V_0}{3\pi} \left[\sin^3 \frac{\pi x}{L} \right]_0^{L/2} \\
&= \frac{4V_0}{3\pi}
\end{aligned}$$

The probability amplitude for transitioning is

$$\begin{aligned}
a_{1\to 2}(T) &= -\frac{i}{\hbar} \int_0^T \frac{4V_0}{3\pi} e^{i(E_2 - E_1)t/\hbar} \, dt \\
&= -\frac{4V_0}{3\pi(E_2 - E_1)} e^{i(E_2 - E_1)T}
\end{aligned}$$

Taking the norm-squared, we get

$$P_{1\to 2} = \frac{16}{9\pi^2} \left(\frac{V_0}{E_2 - E_1} \right)^2$$

(b) Since the potential is changing slowly, then from the adiabatic theorem, we can say that at time T, the particle is in the ground state of the potential with $V(x) = \overline{V}$ for $0 < x < L/2$ and zero for $L/2 < x < L$.

But, given that $\overline{V} \gg E_1$ and therefore also much greater than the ground state energy, we can make an approximation that $\overline{V} \to \infty$, leaving us with an infinite potential well with half the width. The ground state of this potential is clearly

$$\psi_1^T(x) = \sqrt{\frac{4}{L}} \sin \frac{2\pi}{L} \left(x - \frac{L}{2} \right)$$

in the region $L/2 < x < L$ and zero everywhere else.

At time T, the potential immediately reverts back to the original, and the probability amplitude of finding the particle in the first excited state of the original potential is

$$\begin{aligned}
\langle \psi_1^T | \psi_2 \rangle &= \int_0^L \psi_1^{T*}(x) \psi_2(x) \, dx \\
&= \int_{L/2}^L \frac{\sqrt{8}}{L} \sin \left[\frac{2\pi}{L} \left(x - \frac{L}{2} \right) \right] \sin \frac{2\pi x}{L} \, dx \\
&= \frac{2\sqrt{2}}{L} \int_{L/2}^L \left(\sin \frac{2\pi x}{L} \cos \pi - \cos \frac{2\pi x}{L} \sin \pi \right) \sin \frac{2\pi x}{L} \, dx \\
&= -\frac{\sqrt{2}}{L} \int_{L/2}^L 1 - \cos \frac{4\pi x}{L} \, dx \\
&= -\frac{\sqrt{2}}{2}
\end{aligned}$$

So, the probability of finding the particle in the first excited state is

$$P = |\langle \psi_1^T | \psi_2 \rangle|^2 = \frac{1}{2}$$

Of course, this is only an approximation because we did not use the exact ground state of the potential at time T. Since the $\overline{V}$ is so large compared to the energy levels of interest, the wave function is actually a rapidly decaying exponential in this region. We would then have to match boundary conditions at $x = L/2$ and solve a transcendental equation to obtain the correct wave numbers and energy levels. Who has time for that???? So let's just take the easy route and say that to a good approximation, we can take $\overline{V}$ to infinity and the wave function is zero at $x = L/2$.

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